From the solution of the Tsarev system to the solution of the Whitham equations

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Abstract

We study the Cauchy problem for the Whitham modulation equations for monotone increasing smooth initial data. The Whitham equations are a collection of one-dimensional quasi-linear hyperbolic systems. This collection of systems is enumerated by the genus g=0,1,2, ... of the corresponding hyperelliptic Riemann surface. Each of these systems can be integrated by the so called hodograph transform introduced by Tsarev. A key step in the integration process is the solution of the Tsarev linear overdetermined system. For each $g > 0$, we construct the unique solution of the Tsarev system, which matches the genus $g+1$ and $g-1$ solutions on the transition boundaries. Next we characterize initial data such that the solution of the Whitham equations has genus $g \le N$, $N > 0$, for all real $t \ge 0$ and x.

1 Introduction

The Whitham equations are a collection of quasi-linear hyperbolic systems of the form [1],[2],[3]

$$\frac{\partial u_i}{\partial t} - \lambda_i(u_1, u_2, \dots, u_{2g+1}) \frac{\partial u_i}{\partial x} = 0, \quad x, t, u_i \in \mathbb{R}, \quad i = 1, ..., 2g+1, \quad g = 0, 1, 2, \dots, \tag{1.1}$$

with the ordering $u_1 > u_2 > \cdots > u_{2g+1}$. For a given g the system (1.1) is called g-phase Whitham equations. Because of the diagonal form of systems (1.1) the dependent variables $u_1 > u_2 > \cdots > u_{2g+1}$ are called Riemann invariants. For $g > 0$ the speeds $\lambda_i(u_1, u_2, \dots, u_{2g+1})$, $i = 1, 2, \dots, 2g+1$, depend

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through $u_1, \ldots, u_{2g+1}$ on complete hyperelliptic integrals of genus g. For this reason the g-phase system is also called genus g system. The zero-phase Whitham equation has the form

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} = 0\,, \tag{1.2}$$

where we use the notation $u_1 = u$. In this paper we study the initial value problem of the Whitham equations for monotone increasing smooth (C^∞) initial data $u(x, t = 0) = u_0(x)$, where the range of $u_0(x)$, $x \in \mathbb{R}$, is the interval (a, b) and $-\infty \le a < b \le +\infty$.
The initial value problem consists of the following. We consider the evolution on the $x - u$ plane of the initial curve $u(x, t = 0) = u_0(x)$ according to the zero-phase equation (1.2). The solution $u(x, t)$ of (1.2) with the initial data $u_0(x)$ is given by the characteristic equation

$$x = -6tu + f(u) \tag{1.3}$$

where $f(u)$ is the inverse function of $u_0(x)$. The solution $u(x, t)$ in (1.3) is globally well defined only for $0 \le t < t_0$, where $t_0 = \frac{1}{6}\min_{u\in\mathbb{R}}[f'(u)]$ is the time of gradient catastrophe of (1.3). Near the point of gradient catastrophe and for a short time $t > t_0$, the evolving curve is given by a multivalued function with three branches $b > u_1(x, t) > u_2(x, t) > u_3(x, t) > a$, which evolve according to the one-phase Whitham equations.

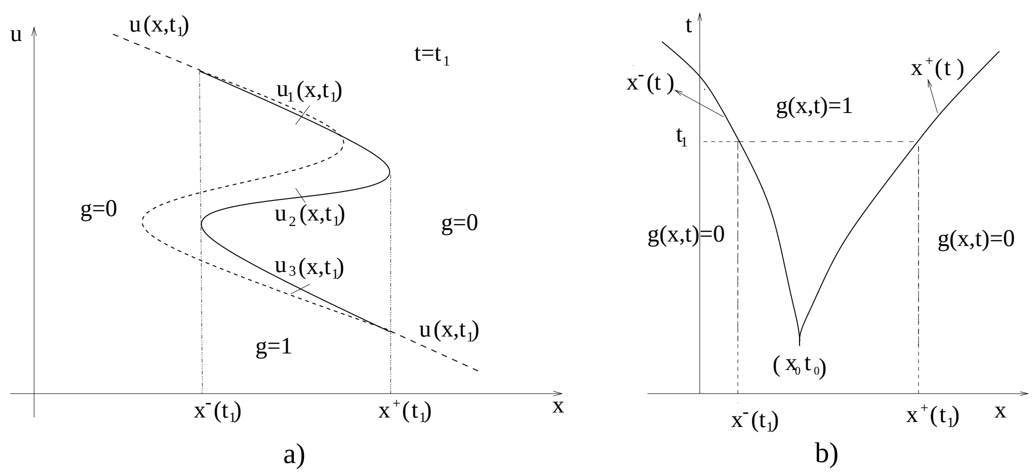

Figure 1.1: on picture a) the dashed line represents the formal solution of the zero-phase equation the continuous line represents the solution of the one-phase equations. The solution $(u_1(x, t)\,,\, u_2(x, t), u_3(x, t))$ of the one-phase equations and the position of the boundaries $x^-(t)$, $x^+(t)$ are to be determined from the condition $u(x^-(t), t) = u_1(x^-(t), t)$, $u_x(x^-(t), t) = u_{1x}(x^-(t), t)$, $u(x^+(t), t) = u_3(x^+(t), t)$, $u_x(x^+(t), t) = u_{3x}(x^+(t), t)$, where $u(x, t)$ is the solution of the zero-phase equation. Picture b) represents the functions $x^-(t)$ and $x^+(t)$ on the $x - t$ plane.

Outside the multivalued region the solution is given by the zero-phase solution $u(x,t)$ defined in (1.3). On the phase transition boundary the zero-phase solution and the one-phase solution are attached C^1-smoothly (see Fig1.1).
Since the Whitham equations are hyperbolic, other points of gradient catastrophe can appear in the branches $u_1(x,t) > u_2(x,t) > u_3(x,t)$ themselves or in $u(x,t)$.

In general, for $t > t_0$, the evolving curve is given by a multivalued function with and odd number of branches $b > u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t) > a$, $g \geq 0$. These branches evolve according to the g-phase Whitham equations. The g-phase solutions for *different* g must be glued together in order to produce a C^1-smooth curve in the (x,u) plane evolving smoothly with t (see Fig. 1.2). The initial value problem of the Whitham equations is to determine, for almost all $t > 0$ and x, the phase $g(x,t) \geq 0$ and the corresponding branches $u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t)$ from the initial data $x = f(u)|_{t=0}$. For example for the initial data $x = x_0 + 6t_0\, u + (u-u_0)^3$, $t_0 \geq 0$,

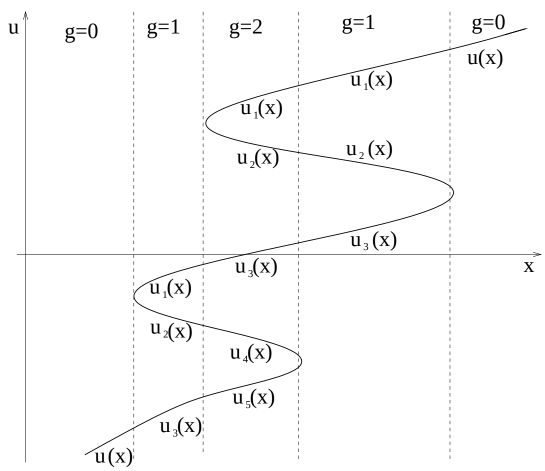

Figure 1.2: the plot on the $x-u$ plane of the evolving multivalued curve for fixed $t > 0$.

the solution of the Whitham equations is of genus at most equal to one [12][6]. It is of genus one inside the cusp $-12\sqrt{3}(t-t_0)^{\frac{3}{2}} < x - x_0 < \frac{4}{3}\sqrt{\frac{5}{3}}(t-t_0)^{\frac{3}{2}}$ [13]; it is of genus zero outside this cusp. The point $x = x_0, t = t_0$ is the point of gradient catastrophe of the zero-phase solution. The curve $x^-(t) = x_0 - 12\sqrt{3}(t-t_0)^{\frac{3}{2}}$, $t > t_0$, describes the locus of points where $u_1(x,t) = u_2(x,t)$. The curve $x^+(t) = x_0 + \frac{4}{3}\sqrt{\frac{5}{3}}(t-t_0)^{\frac{3}{2}}$, $t > t_0$, describes the locus of points where $u_2(x,t) = u_3(x,t)$. For generic initial data it is not known whether the genus of the solution of the Whitham equations is bounded. We say that the Cauchy problem for the Whitham equations has a global solution if the genus $g < \infty$ for all x and $t \geq 0$.
Using the geometric-Hamiltonian structure [4] of the Whitham equations, Tsarev [5] showed that these equations can be locally integrated by a generalization of the method of characteristic. Namely he proved that if the functions $w_i = w_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, \ldots, 2g+1$, solve the linear over-

determined system

$$\frac{\partial w_i}{\partial u_j} = \frac{1}{\lambda_i - \lambda_j}\frac{\partial \lambda_i}{\partial u_j}[w_i - w_j], \quad i,j = 1,2,\ldots,2g+1, \;\; i \neq j, \tag{1.4}$$

where $\lambda_i = \lambda_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, \ldots, 2g+1$, are the speeds in (1.1), then the solution $\vec{u}(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_{2g+1}(x,t))$ of the so called hodograph transform

$$x = -\lambda_i(\vec{u})\, t + w_i(\vec{u}) \quad i = 1, \ldots, 2g+1\,, \tag{1.5}$$

satisfies system (1.1). Conversely, any solution $(u_1(x,t), u_2(x,t), \ldots, u_{2g+1}(x,t))$ of (1.1) can be obtained in this way.

Tsarev theorem relies on two factors:

a) the existence of a solution of the linear over-determined system (1.4);

b) the existence of a real solutions $u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t)$ of the hodograph transform (1.5).

In this paper we solve completely problem a), namely we build for any smooth monotone increasing initial data $x = f(u)|_{t=0}$ the solution of system (1.4) for any $g \geq 0$. The solution $w_i(\vec{u})$, $i = 1, \ldots, 2g+1$, of genus g satisfies some natural boundary conditions which guarantee its uniqueness.

Regarding problem b) we characterize initial data such that the solution of the Whitham equations exists only for $g \leq N$ where N is a positive integer. Namely we show that if the initial data satisfies the condition

$$\frac{d^{2N+1}}{du^{2N+1}} f(u) := f^{(2N+1)}(u) > 0, \quad 1 \leq N \in \mathbb{N} \tag{1.6}$$

for all real u belonging to the domain of f except one point, then the solution of the Whitham equations has genus at most N for any x and $t \geq 0$. For $N = 1$ this result has already been proved by Tian [6].

The investigation of the initial value problem of the Whitham equations was initiated by Gurevich and Pitaevskii [7]. In the case $g \leq 1$ they solved the initial value problem of system (1.1) for step-like initial data and studied numerically the case of cubic initial data.

The initial value problem of the Whitham equations was deeply studied by Tian. In [6] he constructed the general solution of the Tsarev system (1.4) for $0 \leq g \leq 1$ and for smooth monotone increasing initial data. He also proved the solvability of the hodograph transform (1.5) for $0 \leq g \leq 1$. In [8] he obtained the solution of the Tsarev system for $g > 0$ for polynomial initial data.

The equations (1.1) were found by Whitham [1] in the single phase case $g = 1$ and more generally by Flaschka, Forest and McLaughlin [2] in the multi-phase case. The Whitham equations were also found in [3] when studying the zero dispersion limit of the Korteweg de Vries equation. The hyperbolic nature of the equations was found by Levermore [9].

This paper is organized as follows.

In Sec. 2 we give some background about Abelian differentials on hyperelliptic Riemann surfaces.

In Sec. 3 we describe the Whitham equations.
In Sec. 4 we build the solution of the Tsarev system (1.4) for smooth monotone increasing initial data.
In Sec. 5 we show that under the hypothesis (1.6) the hodograph transform is solvable for $g \le N$.
In Sec. 6 we draw the conclusions.

2 Riemann surfaces and Abelian differentials: notations and definitions

Let

$$\mathcal{S}_g := \left\{ P = (r, \mu),\ \mu^2 = \prod_{j=1}^{2g+1} (r - u_j) \right\} , \quad u_1 > u_2 > \cdots > u_{2g+1} , \quad u_i \in \mathbb{R} , \tag{2.1}$$

be the hyperelliptic Riemann surface of genus $g \ge 0$. We shall use the standard representation of $\mathcal{S}_g$ as a two-sheeted covering of $C\mathbb{P}^1$ with cuts along the intervals

$$[u_{2k}, u_{2k-1}], \quad k = 1, \ldots, g+1, \quad u_{2g+2} = -\infty .$$

We choose the basis $\{\alpha_j, \beta_j\}_{j=1}^{g}$ of the homology group $H_1(\Gamma_g)$ so that α_j lies fully on the upper sheet and encircles clockwise the interval $[u_{2j}, u_{2j-1}]$, $j = 1, \ldots, g$, while β_j emerges on the upper sheet on the cut $[u_{2j}, u_{2j-1}]$, passes anti-clockwise to the lower sheet trough the cut $(-\infty, u_{2g+1}]$ and return to the initial point through the lower sheet.
The one-forms that are analytic on the closed Riemann surface $\mathcal{S}_g$ except for a finite number of points are called Abelian differentials.
We define on $\mathcal{S}_g$ the following differentials [10]:
1) The canonical basis of holomorphic one-forms or Abelian differentials of the first kind $\phi_1, \phi_2 \ldots \phi_g$:

$$\phi_k(r) = \frac{r^{g-1}\gamma_1^k + r^{g-2}\gamma_2^k + \cdots + \gamma_g^k}{\mu(r)} dr , \quad k = 1, \ldots, g . \tag{2.2}$$

The constants γ_i^k are uniquely determined by the normalization conditions

$$\int_{\alpha_j} \phi_k = \delta_{jk} , \quad j, k = 1, \ldots, g. \tag{2.3}$$

We remark that an holomorphic differential having all its α-periods equal to zero is identically zero [10].
2) The set σ_k^g, $k \ge 0$, $g \ge 0$, of Abelian differentials of the second kind with a pole of order $2k+2$ at infinity, with asymptotic behavior

$$\sigma_k^g(r) = \left[r^{k-\frac{1}{2}} + O(r^{-\frac{3}{2}}) \right] dr \quad \text{for large } r \tag{2.4}$$

and normalized by the condition

$$\int_{\alpha_j} \sigma_k^g = 0, \quad j = 1, \dots, g\,. \tag{2.5}$$

We use the notation

$$\sigma_0^g(r) = dp^g(r)\,, \qquad 12\sigma_1^g(r) = dq^g(r) \quad g \geq 0. \tag{2.6}$$

In literature the differentials $dp^g(r)$ and $dq^g(r)$ are called quasi-momentum and quasi-energy respectively [4]. The explicit formula for the differentials σ_k^g, $k \geq 0$, is given by the expression

$$\sigma_k^g(r) = \frac{P_k^g(r)}{\mu(r)} dr\,, \quad P_k^g(r) = r^{g+k} + a_1^k r^{g+k-1} + a_2^k r^{g+k-2} \cdots + a_{g+k}^k\,, \tag{2.7}$$

where the coefficients $a_i^k = a_i^k(\vec{u})$, $\vec{u} = (u_1, u_2, \dots, u_{2g+1})$, $i = 1, \dots, g+k$, are uniquely determined by (2.4) and (2.5).
3) The Abelian differential of the third kind $\omega_{qq_0}(r)$ with first order poles at the points $Q = (q, \mu(q))$ and $Q_0 = (q_0, \mu(q_0))$ with residues ± 1 respectively. Its periods are normalized by the relation

$$\int_{\alpha_j} \omega_{qq_0}(r) = 0, \quad j = 1, \dots, g\,. \tag{2.8}$$

2.1 Riemann bilinear relations

Let ω_1 and ω_2 be two Abelian differentials on the Riemann surface $\mathcal{S}_g$. If all the residues of ω_1 and ω_2 are equal to zero, then the integrals $d^{-1}\omega_1$ and $d^{-1}\omega_2$ do not have logarithm singularities on $\mathcal{S}_g$. If the differential ω_1 has non zero residue, then its integral has logarithm singularities. Let s be the path connecting the singular points of $d^{-1}\omega_1$. We have the following relation.

$$\sum_{j=1}^{g} \left[\int_{\alpha_j} \omega_1 \int_{\beta_j} \omega_2 - \int_{\alpha_j} \omega_2 \int_{\beta_j} \omega_1 \right] + \int_s \Delta(d^{-1}\omega_1)\omega_2 = 2\pi i \sum_{\mathcal{S}_g - s} \mathrm{Res}[(d^{-1}\omega_1)\omega_2]\,, \tag{2.9}$$

where $\Delta(d^{-1}\omega_1)$ is the difference of the values of $d^{-1}\omega_1$ on the two sides of the cut s and the quantity $\sum_{\mathcal{S}_g - s} \mathrm{Res}[(d^{-1}\omega_1)\omega_2]$ is the sum of the residues of the differential $(d^{-1}\omega_1)\omega_2$ on the cut surface $\mathcal{S}_g - s$. This formula is known as the Riemann bilinear period relation [11].
Assuming $\omega_1 = \omega_{qq_0}$ and $\omega_2 = \phi_k$ in (2.9) we obtain

$$\int_{\beta_k} \omega_{qq_0} = 2\pi i \int_{q_0}^{q} \phi_k\,, \quad k = 1, \dots, g\,. \tag{2.10}$$

Assuming $\omega_1 = \omega_{qq_0}$ and $\omega_2 = \omega_{pp_0}$ in (2.9) we obtain

$$\int_{p_0}^{p} \omega_{qq_0} = \int_{q_0}^{q} \omega_{pp_0}\,. \tag{2.11}$$

Differentiating with respect to p and q the above expression we obtain the identity

$$d_q[\omega_{qq_0}(p)] = d_p[\omega_{pp_0}(q)]\,, \tag{2.12}$$

where d_q and d_p denote differentiation with respect to q and p respectively. From the expression (2.11) it follows that $\omega_{qq_0}(r)$ is a many-value analytic function of the variable q. The many-value character of $\omega_{qq_0}(r)$ as a function of q can be described by the equations

$$\int_{\alpha_k} d_q[\omega_{qq_0}(r)] = 0\,, \quad \int_{\beta_k} d_q[\omega_{qq_0}(r)] = 2\pi i \phi_k(r), \quad k = 1,\dots,g\,, \tag{2.13}$$

In the following we mainly use the normalized differential $\omega_z^g(r)$ which has simple poles at the points $Q^{\pm}(z) = (z, \pm\mu(z))$ with residue ± 1 respectively.

The differential $\omega_z^g(r)$ is explicitly given by the expression

$$\omega_z^g(r) = \frac{dr}{\mu(r)}\frac{\mu(z)}{r-z} - \sum_{k=1}^{g}\phi_k(r)\int_{\alpha_k}\frac{dt}{\mu(t)}\frac{\mu(z)}{t-z}\,, \tag{2.14}$$

where $\phi_k(r)$, $k = 1,\dots,g$, is the normalized basis of holomorphic differentials. $\omega_z^g(r)$ as a function of z, is an Abelian integral having poles of first order at the points $Q^{\pm}(r) = (r, \pm\mu(r))$. The periods of this integral are obtained from the relations (2.13)

$$\int_{\alpha_j} d_z[\omega_z^g(r)] = 0, \quad \int_{\beta_j} d_z[\omega_z^g(r)] = 4\pi i\phi_k(r)\,, \quad j = 1,\dots g\,. \tag{2.15}$$

We apply the Riemann bilinear relation (2.9) to the differentials $\sigma_m^g(r)$ and $\omega_z^g(r)$ getting

$$\begin{aligned}\int_{Q^-(z)}^{Q^+(z)}\sigma_m^g(\xi) &= -\operatorname*{Res}_{r=\infty}[\omega_z^g(r)d^{-1}\sigma_m^g(r)], \quad m = 0,\dots,g,\\ &= -\frac{4}{2m+1}\left(-\mu(z)\epsilon_{mg} + \sum_{k=1}^{g}\sum_{j=1}^{g}\gamma_j^k\Gamma_{m+1-j}\int_{\alpha_k}\frac{dt}{\mu(t)}\frac{\mu(z)}{t-z}\right)\,.\end{aligned} \tag{2.16}$$

In the above formula $\epsilon_{mg} = 1$ for $m = g$ and zero otherwise, the coefficients γ_j^k have been defined in (2.2) and the Γ_l's are the coefficients of the expansion for $\xi \to \infty$ of

$$\frac{1}{\mu(\xi)} = \xi^{-g-\frac{1}{2}}(\Gamma_0 + \frac{\Gamma_1}{\xi} + \frac{\Gamma_2}{\xi^2} + \dots + \frac{\Gamma_l}{\xi^l} + \dots). \tag{2.17}$$

We define $\Gamma_k = 0$ for $k < 0$.

Inverting (2.16) and introducing the quantities $N_j(z,\vec{u}) = \sum_{k=1}^{g}\gamma_j^k\int_{\alpha_k}\frac{dt}{\mu(t)}\frac{\mu(z)}{t-z}$ we obtain

$$\begin{pmatrix} N_1(z,\vec{u}) \\ N_2(z,\vec{u}) \\ \dots \\ N_g(z,\vec{u}) \\ -\mu(z) \end{pmatrix} = \begin{pmatrix} \tilde{\Gamma}_0 & 0 & 0 & \dots & 0 \\ \tilde{\Gamma}_1 & \tilde{\Gamma}_0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \tilde{\Gamma}_{g-1} & \tilde{\Gamma}_{g-2} & \dots & \tilde{\Gamma}_0 & 0 \\ \tilde{\Gamma}_g & \tilde{\Gamma}_{g-1} & \dots & \tilde{\Gamma}_1 & \tilde{\Gamma}_0 \end{pmatrix} \begin{pmatrix} -\frac{1}{4}\int_{Q^-(z)}^{Q^+(z)}\sigma_0^g(\xi) \\ -\frac{3}{4}\int_{Q^-(z)}^{Q^+(z)}\sigma_1^g(\xi) \\ \dots \\ -\frac{2g-1}{4}\int_{Q^-(z)}^{Q^+(z)}\sigma_{g-1}^g(\xi) \\ -\frac{2g+1}{4}\int_{Q^-(z)}^{Q^+(z)}\sigma_g^g(\xi) \end{pmatrix} \tag{2.18}$$

where the $\tilde{\Gamma}_k$'s are the coefficients of the expansion for $\xi \to \infty$ of

$$\mu(\xi) = \xi^{g+\frac{1}{2}}(\tilde{\Gamma}_0 + \frac{\tilde{\Gamma}_1}{\xi} + \frac{\tilde{\Gamma}_2}{\xi^2} + \cdots + \frac{\tilde{\Gamma}_l}{\xi^l} + \dots). \tag{2.19}$$

Using (2.18) the differential $\omega_z^g(r)$ turns out to be given by the relation

$$\omega_z^g(r) = \frac{dr}{\mu(r)}\frac{\mu(z)}{r-z} - \sum_{k=1}^{g} N_k(z,\vec{u})\frac{r^{g-k}}{\mu(r)}dr\,. \tag{2.20}$$

We remark that $\omega_z^g(r)$ is a multivalued function of z and it is regular at infinity.
From the relation (2.18) we obtain the identity which will be useful later

$$\mu(z) = \frac{1}{4}\sum_{k=1}^{g+1}(2k-1)\tilde{\Gamma}_{g+1-k}\int_{Q^-(z)}^{Q^+(z)} \sigma_{k-1}^g(\xi). \tag{2.21}$$

The next proposition is also important for our subsequent considerations.

Proposition 2.1 *The Abelian differentials of the second kind $\sigma_k^g(r)$, $k \geq 0$, defined in (2.4) satisfy the relations*

$$\sigma_k^g(r) = \frac{1}{2}\operatorname*{Res}_{z=\infty}\left[\omega_z^g(r)\, z^{k-\frac{1}{2}}dz\right] = -\frac{1}{2k+1}d_r \operatorname*{Res}_{z=\infty}\left[\omega_r^g(z)\, z^{k+\frac{1}{2}}\right]\,, \tag{2.22}$$

where $\omega_z^g(r)$ has been defined in (2.14), $\omega_r^g(z)$ is the normalized Abelian differential of the third kind with simple poles at the points $Q^{\pm}(r) = (r, \pm\mu(r))$ with residue ± 1 respectively and d_r denotes differentiation with respect to r.

Proof: the differential $\mathrm{Res}_{z=\infty}\left[\omega_z^g(r)\, z^{k-\frac{1}{2}}dz\right]$ is normalized because $\omega_z^g(r)$ is a normalized differential. Furthermore

$$\operatorname*{Res}_{z=\infty}\left[\omega_z^g(r)\, z^{k-\frac{1}{2}}dz\right] = r^{k-\frac{1}{2}}dr + O(r^{-\frac{3}{2}})dr \quad \text{for } r \to \infty.$$

Therefore $\mathrm{Res}_{z=\infty}\left[\omega_z^g(r)\, z^{k-\frac{1}{2}}dz\right]$ coincides with the normalized Abelian differential of the second kind $\sigma_k^g(r)$. For proving the second equality in (2.22) we consider the integral in the z variable

$$0 = \oint_{C_\infty} d_z(\omega_z^g(r)\, z^{k+\frac{1}{2}}) = \oint_{C_\infty} z^{k+\frac{1}{2}}(d_z\omega_z^g(r)) + \oint_{C_\infty}(k-\frac{1}{2})(\omega_z^g(r)\, z^{k+\frac{1}{2}})\,,$$

where C_∞ is a close contour around the point at infinity. Substituting the identity $d_z\omega_z^g(r) = d_r\omega_r^g(z)$ in the right hand side of the above relation we obtain the second relation in (2.22). $\square$

3 Preliminaries on the theory of the Whitham equations

The speeds $\lambda_i(u_1, u_2, \dots, u_{2g+1})$ of the g-phase Whitham equations (1.1) are given by the ratio [1],[2]:

$$\lambda_i(\vec{u}) = \left.\frac{dq^g(r)}{dp^g(r)}\right|_{r=u_i}\,, \quad i = 1, 2, \dots, 2g+1\,, \tag{3.1}$$

where $dp^g(r)$ and $dq^g(r)$ have been defined in (2.6). In the case $g=0$

$$dp^0(r) = \frac{dr}{\sqrt{r-u}}\,, \quad dq^0(r) = \frac{12r-6u}{\sqrt{r-u}}dr\,, \tag{3.2}$$

so that one obtains the zero-phase Whitham equation (1.2).

For monotone increasing smooth initial data $x = f(u)|_{t=0}$, the solution of the zero-phase equation (1.2) is obtained by the method of characteristic [1] and is given by the expression

$$x = -6tu + f(u)\,. \tag{3.3}$$

The zero-phase solution is globally well defined only for $0 \le t < t_0$ where $t_0 = \frac{1}{6}\min_{u\in\mathbb{R}}[f'(u)]$ is the time of gradient catastrophe of the solution (3.3). The breaking is cause by an inflection point in the initial data. For $t \ge t_0$ we expect to have single, double and higher phase solutions. For higher genus the Whitham equations can be locally integrated using a generalization of the characteristic equation (3.3). We have the following theorem of Tsarev [5]

Theorem 3.1 *If $w_i(\vec{u})$, $\vec{u} = (u_1, u_2, \ldots, u_{2g+1})$, solves the linear over-determined system*

$$\frac{\partial w_i}{\partial u_j} = a_{ij}(\vec{u})[w_i - w_j], \quad i,j = 1,2,\ldots,2g+1, \;\; i \ne j, \tag{3.4}$$

where

$$a_{ij} = \frac{1}{\lambda_i - \lambda_j}\frac{\partial \lambda_i}{\partial u_j} \quad i,j = 1,2,\ldots,2g+1, \;\; i \ne j, \tag{3.5}$$

then the solution $(u_1(x,t), u_2(x,t), \ldots, u_{2g+1}(x,t))$ of the hodograph transformation

$$x = -\lambda_i(\vec{u})\,t + w_i(\vec{u}) \quad i = 1, \ldots, 2g+1\,, \tag{3.6}$$

satisfies system (1.1). Conversely, any solution $(u_1, u_2, \ldots, u_{2g+1})$ of (1.1) can be obtained in this way.

To guarantee that the $g-$phase solutions for different g are attached continuously, the following natural boundary conditions must be imposed on $w_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, \ldots, 2g+1$.
When $u_l = u_{l+1}$, $1 \le l \le 2g$,

$$w_l^g(u_1, \ldots, u_l, u_l, \ldots, u_{2g+1}) = w_{l+1}^g(u_1, \ldots, u_l, u_l, \ldots, u_{2g+1}) \tag{3.7}$$

and for $1 \le i \le 2g+1$, $i \ne l, l+1$

$$w_i^g(u_1, \ldots, u_l, u_l, \ldots, u_{2g+1}) = w_i^{g-1}(u_1, \ldots, \hat{u}_l, \hat{u}_l, \ldots, u_{2g+1}). \tag{3.8}$$

The sup-scripts g and $g-1$ in the w_i's specify the corresponding genus and the hat denotes the variable that have been dropped. When $g=1$ and $u_2 = u_3$ we have that

$$\begin{aligned} w_1(u_1, u_3, u_3) &= f(u_1) \\ w_2(u_1, u_3, u_3) &= w_3(u_1, u_3, u_3)\,, \end{aligned} \tag{3.9}$$

where $f(u)$ is the initial data. Similar conditions hold true when $u_1 = u_2$, namely

$$\begin{aligned} &w_3(u_1, u_1, u_3) = f(u_3) \\ &w_1(u_1, u_1, u_3) = w_2(u_1, u_1, u_3). \end{aligned} \tag{3.10}$$

We remark that the $\lambda_i(\vec{u})$'s satisfy the boundary conditions (3.7-3.8) and for $g = 1$ we have [8]

$$\lambda_1(u_1, u_3, u_3) = -6u_1, \quad \lambda_2(u_1, u_3, u_3) = \lambda_3(u_1, u_3, u_3)\,,$$

and

$$\lambda_3(u_1, u_1, u_3) = -6u_3, \quad \lambda_1(u_1, u_1, u_3) = \lambda_2(u_1, u_1, u_3).$$

The solution of the boundary value problem (3.4), (3.7-3.10) has been obtained in [8, 15] for monotone increasing analytic initial data of the form

$$x = f_a(u) = c_0 + c_1 u + \cdots + c_k u^k + \ldots \tag{3.11}$$

where we assume that only a finite number of c_k is different from zero. For such initial data the $w_i(\vec{u})$'s which satisfy (3.4) and the boundary conditions (3.7-3.10) are given by the expression [8]

$$w_i(\vec{u}) = \left.\frac{ds^g(r)}{dp^g(r)}\right|_{r=u_i}, \quad i = 1, \ldots, 2g+1. \tag{3.12}$$

The differential ds^g in (3.12) is given by

$$ds^g(r) = \sum_{k=0}^{\infty} \frac{2^k k!}{(2k-1)!!} c_k \sigma_k^g(r)\,, \tag{3.13}$$

and the differentials $\sigma_k^g(r),\ k \geq 0$ have been defined in (2.7).
The solution (3.6) of the g-phase Whitham equations can also be written in an equivalent algebro-geometric form [8, 15] namely

$$(-x dp^g(r) - t dq^g(r) + ds^g(r))\,|_{r=u_i} = 0\,, \quad i = 1, 2 \ldots, 2g+1\,, \tag{3.14}$$

where dp^g, dq^g and ds^g have been defined in (2.6) and (3.13) respectively.
The solution $u_1 > u_2 > \cdots > u_{2g+1}$ of the g-phase Whitham equations (1.1) is implicitly defined as a function of x and t by the equations (3.6) or (3.14). The solution is uniquely defined only for those x and t such that the functions $u_i(x, t)$ are real and $\partial_x u_i(x, t)$, $i = 1, \ldots, 2g+1$, are not vanishing.
One of the problems in the theory of the Whitham equations is to determine when (3.6) or (3.14) are solvable for real $u_1, \ldots, u_{2g+1}$ as functions of x and t. This problem has been solved by Tian for $g \leq 1$.

Theorem 3.2 *[6] Consider a monotone increasing initial data $x = f(u)|_{t=0}$. Suppose that u^* is the inflection point of $f(u)$ that causes the breaking of the zero-phase solution (1.2) at $x = x_0,\ t = t_0$. Let be $f'''(u) > 0$ in a deleted neighborhood of $u = u^*$. Then the one-phase Whitham equations has a solution $u_1 > u_2 > u_3$ within a cusp in the $x - t$ plane for a short time after the breaking time of the zero-phase solution. Furthermore this solution satisfies the boundary conditions (3.9) and (3.10) on the cusp. If the initial data satisfies the condition $f'''(u) > 0$ for all u except $u = u^*$, then the solution of the Whitham equations exists for all $t > 0$. The solution is of genus one inside the cusp $x^-(t) < x < x^+(t)$, $t > t_0$, where $x^-(t) < x^+(t)$ are two real functions satisfying the condition $x^-(t_0) = x^+(t_0) = x_0$. The solution is of genus zero outside the cusp $x^-(t) < x < x^+(t)$, $t > t_0$.*

4 Solution of Tsarev system

In this section we build the solution of the boundary value problem (3.4), (3.7-3.10) for monotone smooth initial data and we show that the solution obtained is unique. We consider initial data of the form $x = f(u)|_{t=0}$ where $f(u)$ is a monotone increasing function. The domain of f is the interval (a, b) where $-\infty \le a < b \le +\infty$, and the range of f is the real line $(-\infty, +\infty)$.
In order to obtain the solution of the boundary value problem (3.4), (3.7-3.10) we need the following technical lemma.

Lemma 4.1 *The differential $ds^g(r)$ defined in (3.13) can be written in the form*

$$ds^g(r) = 2\mu(r)\left(\partial_r \Psi^g(r;\vec{u}) + \sum_{k=1}^{2g+1} \partial_{u_k}\Psi^g(r;\vec{u})\right) dr + \frac{R^g(r)}{\mu(r)} dr, \tag{4.1}$$

where

$$\Psi^g(r;\vec{u}) = -\operatorname*{Res}_{z=\infty}\left[\frac{\mathcal{F}(z)dz}{2\mu(z)(z-r)}\right], \quad q_k(\vec{u}) = -\operatorname*{Res}_{z=\infty}\left[\frac{z^{g-k}\mathcal{F}(z)dz}{2\mu(z)}\right],\ k = 1, \dots g, \tag{4.2}$$

$$\mathcal{F}(z) = \int_0^z \frac{f_a(\xi)}{\sqrt{z-\xi}} d\xi, \tag{4.3}$$

$$R^g(r) = 2\sum_{k=1}^{2g+1} \partial_{u_k} q_g(\vec{u}) \prod_{l=1, l\neq k}^{2g+1} (r - u_l) + \sum_{k=1}^{g} q_k(\vec{u}) \sum_{l=1}^{k} (2l-1)\tilde{\Gamma}_{k-l} P^g_{l-1}(r), \tag{4.4}$$

and the polynomials $P^g_l(r)$, $l \ge 0$, have been defined in (2.7), the $\tilde{\Gamma}_k$'s have been defined in (2.19).

Proof: using the second identity in (2.22), we rewrite the differential $ds^g(r)$ defined in (3.13) in the form

$$ds^g(r) = -d_r\left(\operatorname*{Res}_{z=\infty}[\omega^g_r(z)\mathcal{F}(z)]\right), \tag{4.5}$$

where $\omega_r^g(z)$ has been defined in (2.14) and $\mathcal{F}(z)$ is the Abel transform defined in (4.3) of the analytic initial data (3.11). The identity (4.5) can be checked straightforward. Using the explicit expression of $\omega_r^g(z)$ in (2.20) we obtain

$$ds^g(r) = 2d_r(\mu(r)\Psi^g(r;\vec{u})) + \sum_{k=1}^{g} q_k(\vec{u}) \sum_{l=1}^{k} (2l-1)\tilde{\Gamma}_{k-l}\sigma_{l-1}^g(r), \tag{4.6}$$

where $\Psi^g(r;\vec{u})$ and $q_k(\vec{u})$ have been defined in (4.2).
From (4.2) we get the relations

$$\frac{\Psi^g(r;\vec{u})}{r-u_i} - \frac{\Psi^g(u_i;\vec{u})}{r-u_i} = 2\partial_{u_i}\Psi^g(r;\vec{u}), \quad 2\partial_{u_i} q_g(\vec{u}) = \Psi^g(u_i;\vec{u}) \tag{4.7}$$

and for $g = 0$ we define $u_1 = u$ and

$$2\partial_u q_0(u) := \Psi^0(u;u) = f_a(u).$$

Using (4.7) we transform the expression for $ds^g(r)$ in (4.6) to the form (4.1). $\square$

The relation (4.1) enables us to write the quantities $w_i(\vec{u}) = \left.\frac{ds^g(r)}{dp^g(r)}\right|_{r=u_i}, \quad i = 1,\dots,2g+1$, in (3.12) in the form

$$w_i(\vec{u}) = \frac{1}{P_0^g(u_i)} \left[2\partial_{u_i} q_g(\vec{u}) \prod_{l=1, l\neq i}^{2g+1} (u_i - u_l) + \sum_{k=1}^{g} q_k(\vec{u}) \sum_{l=1}^{k} (2l-1)\tilde{\Gamma}_{k-l} P_{l-1}^g(u_i) \right]. \tag{4.8}$$

Observe that in the formula (4.8) all the information on the initial data is contained in the functions $q_k(\vec{u})$. The functions $q_k = q_k(\vec{u})$, $k = 1,\dots,g$, solve the linear over-determined system [8]

$$\begin{aligned} 2(u_i - u_j)\frac{\partial^2 q_k(\vec{u})}{\partial u_i \partial u_j} &= \frac{\partial q_k(\vec{u})}{\partial u_i} - \frac{\partial q_k(\vec{u})}{\partial u_j}, \quad i,j = 1,\dots,2g+1, \\ q_k(\underbrace{u,u,\dots,u}_{2g+1}) &= \frac{2^{g-1}}{(2g-1)!!} u^{-k+\frac{1}{2}} \frac{d^{g-k}}{du} \left(u^{g-\frac{1}{2}} f_a^{(k-1)}(u) \right), \end{aligned} \tag{4.9}$$

where $f_a^{(k-1)}(u)$ is the $(k-1)th$ derivative of the polynomial initial data $f_a(u)$. The function $\Psi^g(r;\vec{u})$ in (4.2) satisfies a similar linear over-determined system.

Theorem 4.2 (First Main Theorem) *Let be $f(u)$ a smooth monotone increasing function with domain (a,b), $-\infty \le a < b \le +\infty$. If $q_k = q_k(u_1, u_2, \dots, u_{2g+1})$, $1 \le k \le g$, is the symmetric solution*

of the linear over-determined system

$$\begin{cases} 2(u_i-u_j)\dfrac{\partial^2 q_k(\vec{u})}{\partial u_i\partial u_j}=\dfrac{\partial q_k(\vec{u})}{\partial u_i}-\dfrac{\partial q_k(\vec{u})}{\partial u_j}, \quad i\neq j,\ i,j=1,\dots,2g+1,\ \ g>0 \\ \\ q_k(\underbrace{u,u,\dots,u}_{2g+1})=F_k(u) \\ \\ F_k(u)=\dfrac{2^{(g-1)}}{(2g-1)!!}u^{-k+\frac{1}{2}}\dfrac{d^{g-k}}{du^{g-k}}\left(u^{g-\frac{1}{2}}f^{(k-1)}(u)\right), \end{cases} \tag{4.10}$$

with the ordering $u_1>u_2>\dots>u_{2g+1}$*, then*

$$w_i(\vec{u})=\frac{1}{P_0^g(u_i)}\left[2\partial_{u_i}q_g(\vec{u})\prod_{l=1,l\neq i}^{2g+1}(u_i-u_l)+\sum_{k=1}^{g}q_k(\vec{u})\sum_{l=1}^{k}(2l-1)\tilde{\Gamma}_{k-l}P_{l-1}^g(u_i)\right], \quad i=1,\dots,2g+1, \tag{4.11}$$

solves the boundary value problem (3.4), (3.7-3.10). Conversely every solution of (3.4), (3.7-3.10) can be obtained in this way.

Before proving the theorem we show how to obtain the solution of system (4.10) for generic smooth initial data. We follow the procedure in [8]. We start with the following lemma.

Lemma 4.3 *[8] The system*

$$\begin{cases} 2(z-y)p_{zy} &= p_z-\rho p_y, \quad \rho>0 \\ p(z,z) &= s(z) \end{cases} \tag{4.12}$$

has, for any smooth initial data $s(z)$ *one and only one solution. Moreover, the solution can be written explicitly*

$$p(z,y)=\frac{1}{C_\rho}\int_{-1}^{1}\frac{s(\frac{1+\mu}{2}z+\frac{1-\mu}{2}y)}{\sqrt{1-\mu}}(1+\mu)^{\frac{\rho}{2}-1}d\mu \tag{4.13}$$

where

$$C_\rho=\int_{-1}^{1}\frac{(1+\mu)^{\frac{\rho}{2}-1}}{\sqrt{1-\mu}}d\mu\,. \tag{4.14}$$

Using the above lemma, the linear over-determined systems (4.10) can be integrated for any smooth initial data in the following way. Suppose that $q_k(u_1,u_2,\dots,u_{2g+1})$ is a solution of (4.10).

Clearly $A_k(u_1,u_{2g+1})=q_k(\underbrace{u_1,u_1,\dots,u_1}_{2g},u_{2g+1})$ satisfies

$$\begin{aligned} 2(u_1-u_{2g+1})\frac{\partial^2 A_k}{\partial u_1\partial u_{2g+1}} &= \frac{\partial A_k}{\partial u_1}-2g\frac{\partial A_k}{\partial u_{2g+1}} \\ A_k(u,u) &= F_k(u) \end{aligned} \tag{4.15}$$

which by lemma 4.3 implies that

$$A_k(u_1,u_{2g+1}) = \frac{1}{C_{2g}}\int_{-1}^{1}\frac{F_k(\frac{1+\xi_{2g}}{2}u_1+\frac{1-\xi_{2g}}{2}u_{2g+1})}{\sqrt{1-\xi_{2g}}}(1+\xi_{2g})^{g-1}d\xi_{2g}. \tag{4.16}$$

For each fixed u_{2g+1} the function $B_k(u_1,u_{2g},u_{2g+1}) = q_k(\underbrace{u_1,\dots,u_1}_{2g-1},u_{2g},u_{2g+1})$ satisfies

$$\begin{aligned} 2(u_1-u_{2g})\frac{\partial^2 B_k}{\partial u_1\partial u_{2g}} &= \frac{\partial B_k}{\partial u_1}-(2g-1)\frac{\partial B_k}{\partial u_{2g}} \\ B_k(u,u,u_{2g+1}) &= A_k(u,u_{2g+1}) \end{aligned} \tag{4.17}$$

Using again lemma 4.3 we obtain

$$\begin{aligned} B_k(u_1,u_{2g},u_{2g+1}) =&\frac{1}{C_{2g}C_{2g-1}}\int_{-1}^{1}\int_{-1}^{1}d\xi_{2g}d\xi_{2g-1}(1+\xi_{2g})^{g-1}(1+\xi_{2g-1})^{g-\frac{3}{2}}\times \\ &\frac{F_k(\frac{1+\xi_{2g}}{2}(\frac{1+\xi_{2g-1}}{2}u_1+\frac{1+\xi_{2g-1}}{2}u_{2g})\frac{1-\xi_{2g}}{2}u_{2g+1})}{\sqrt{1-\xi_{2g}}\sqrt{1-\xi_{2g-1}}} \end{aligned} \tag{4.18}$$

Going on in the process of integration we obtain the solution $q_k(\vec{u}) = q_k(u_1,u_2,\dots,u_{2g+1})$ of the boundary value problem (4.10) namely

$$\begin{aligned} q_k(\vec{u}) =&\frac{1}{C}\int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1}d\xi_1 d\xi_2\dots d\xi_{2g}(1+\xi_{2g})^{g-1}(1+\xi_{2g-1})^{g-\frac{3}{2}}\dots(1+\xi_3)^{\frac{1}{2}}(1+\xi_1)^{-\frac{1}{2}}\times \\ &\frac{F_k(\frac{1+\xi_{2g}}{2}(\dots(\frac{1+\xi_2}{2}(\frac{1+\xi_1}{2}u_1+\frac{1-\xi_1}{2}u_2)+\frac{1-\xi_2}{2}u_3)+\dots)+\frac{1-\xi_{2g}}{2}u_{2g+1})}{\sqrt{(1-\xi_1)(1-\xi_2)\dots(1-\xi_{2g})}} \end{aligned} \tag{4.19}$$

where $C=\prod_{j=1}^{2g}C_j$ and C_j has been defined in (4.14). When the initial data is of the form (3.11), the expression (4.19) for the $q_k(\vec{u})$'s is equivalent to (4.2).

Theorem 4.4 *The boundary value problem (4.10) has one and only one solution. The solution is symmetric and is given by (4.19).*

Proof: Uniqueness follows from lemma 4.3 and the argument previous to (4.19). The boundary condition (4.10) is clearly satisfied. The symmetry follows from the construction. Indeed in the process of integration we can interchange the role of any of the variable u_i. □

We have the following relations.

Lemma 4.5 *The functions $F_k(u)$ and the solutions $q_k(\vec{u})$, $k=1,\dots,g$, of the boundary value problem (4.10) satisfy the following relations.*

$$\begin{aligned}\partial_u F_k(u) &= \frac{2g+1}{2}F_{k+1}(u) + u\partial_u F_{k+1}(u), \quad k=1,\dots,g-1, \quad g>0\\ \partial_{u_i} q_k(\vec{u}) &= \frac{1}{2}q_{k+1}(\vec{u}) + u_i\partial_{u_i} q_{k+1}(\vec{u}) \quad i=1\dots,2g+1, \quad k=1,\dots,g-1 \quad g>0.\end{aligned} \tag{4.20}$$

Proof of Theorem 4.2 (First Main Theorem).
We consider the non trivial case where $q_k(\vec{u}) \not\equiv 0$, $k=1,\dots g$, and $\partial_{u_j} q_g(\vec{u}) \not\equiv 0$, $j=1,\dots,2g+1$. The proof consists of three parts.

a) *The $w_i(\vec{u})$'s defined in (4.11) satisfy (3.4).*

Using the definition of $w_i(\vec{u})$ in (4.11) we have the following relation

$$\begin{aligned}\partial_{u_j} w_i(\vec{u}) =& 2\frac{\prod_{\substack{l=1\\ l\neq i}}^{2g+1}(u_i-u_l)}{P_0^g(u_i)}\partial_{u_j}\partial_{u_i} q_g(\vec{u}) - 2\frac{\prod_{\substack{l=1\\ l\neq i}}^{2g+1}(u_i-u_l)}{P_0^g(u_i)}\partial_{u_i} q_g(\vec{u})\left(\frac{\partial_{u_j}P_0^g(u_i)}{P_0^g(u_i)} + \frac{1}{u_i-u_j}\right)\\ &+\partial_{u_j}\left(\sum_{l=1}^{g}(2l-1)\frac{P_{l-1}^g(u_i)}{P_0^g(u_i)}\sum_{k=l}^{g} q_k(\vec{u})\tilde{\Gamma}_{k-l}\right), \quad i\neq j, \ \ i,j=1,\dots,2g+1.\end{aligned} \tag{4.21}$$

The following identities hold [8]

$$\frac{\partial}{\partial_{u_j}}\frac{P_k^g(u_i)}{P_0^g(u_i)} = \frac{1}{\lambda_i-\lambda_j}\frac{\partial\lambda_i}{\partial u_j}\left(\frac{P_k^g(u_i)}{P_0^g(u_i)} - \frac{P_k^g(u_j)}{P_0^g(u_j)}\right), \quad i\neq j, i,j=1,\dots,2g+1, \ \ k\geq 1 \tag{4.22}$$

$$\frac{1}{\lambda_i-\lambda_j}\frac{\partial\lambda_i}{\partial u_j} = -\frac{\partial_{u_j}P_0^g(u_i)}{P_0^g(u_i)} - \frac{1}{2}\frac{1}{u_i-u_j}, \quad i\neq j, \ i,j=1,\dots,2g+1, \tag{4.23}$$

where $\lambda_i(\vec{u})$ has been defined in (3.1) and $P_k^g(r)$ has been defined in (2.4).
Using the definition of $\tilde{\Gamma}_k$ in (2.19) it is easy to verify that

$$\frac{\partial\tilde{\Gamma}_k}{\partial u_j} = -\frac{1}{2}\sum_{m=1}^{k-1}\tilde{\Gamma}_{k-m}u_j^{m-1}. \tag{4.24}$$

Applying repeatedly the relations (4.20) we obtain the following expression for $\partial_{u_j} q_k(\vec{u})$:

$$\partial_{u_j} q_k(\vec{u}) = \frac{1}{2}\sum_{m=1}^{g-k} q_{m+k}(\vec{u})u_j^{m-1} + u_j^{g-k}\partial_{u_j} q_g(\vec{u}), \quad k=1,\dots,g-1. \tag{4.25}$$

From (4.10) and (4.22-4.25) we can write $\partial_{u_j} w_i(\vec{u})$ in (4.21) in the form

$$
\begin{aligned}
\partial_{u_j} w_i(\vec{u}) =& \frac{\prod_{l=1,l\neq i}^{2g+1}(u_i-u_l)}{P_0^g(u_i)} \frac{\partial_{u_i} q_g(\vec{u}) - \partial_{u_j} q_g(\vec{u})}{u_i-u_j} \\
& -2\frac{\prod_{l=1,l\neq i}^{2g+1}(u_i-u_l)}{P_0^g(u_i)} \partial_{u_i} q_g(\vec{u}) \left(\frac{\partial_{u_j} P_0^g(u_i)}{P_0^g(u_i)} + \frac{1}{u_i-u_j} \right) \\
& + \sum_{l=1}^{g} (2l-1) \frac{P_{l-1}^g(u_i)}{P_0^g(u_i)} \left(\frac{1}{2} \sum_{k=l}^{g-1} \tilde{\Gamma}_{k-l} \sum_{m=1}^{g-k} q_{m+k}(\vec{u}) u_j^{m-1} + \sum_{k=l}^{g} \tilde{\Gamma}_{k-l} u_j^{g-k} \partial_{u_j} q_g(\vec{u}) \right) \\
& + \sum_{l=1}^{g} (2l-1) \frac{P_{l-1}^g(u_i)}{P_0^g(u_i)} \sum_{k=l}^{g} q_k(\vec{u}) \left(-\frac{1}{2} \sum_{m=1}^{k-l} \tilde{\Gamma}_{k-l-m} u_j^{m-1} \right) \\
& + \sum_{l=1}^{g} (2l-1) \frac{1}{\lambda_i-\lambda_j} \frac{\partial \lambda_i}{\partial u_j} \left(\frac{P_{l-1}^g(u_i)}{P_0^g(u_i)} - \frac{P_{l-1}^g(u_j)}{P_0^g(u_j)} \right) \sum_{k=l}^{g} q_k(\vec{u}) \tilde{\Gamma}_{k-l} \quad i \neq j,\ i,j = 1,\dots,2g+1.
\end{aligned}
$$

Simplifying we obtain

$$
\begin{aligned}
\partial_{u_j} w_i(\vec{u}) =& -\frac{\prod_{l=1,l\neq i}^{2g+1}(u_i-u_l)}{(u_i-u_j)P_0^g(u_i)} \partial_{u_j} q_g(\vec{u}) + 2\frac{\prod_{l=1,l\neq i}^{2g+1}(u_i-u_l)}{P_0^g(u_i)} \partial_{u_i} q_g(\vec{u}) \left(\frac{1}{\lambda_i-\lambda_j} \frac{\partial \lambda_i}{\partial u_j} \right) \\
& + \sum_{l=1}^{g} (2l-1) \frac{P_{l-1}^g(u_i)}{P_0^g(u_i)} \sum_{k=l}^{g} \tilde{\Gamma}_{k-l} u_j^{g-k} \partial_{u_j} q_g(\vec{u}) \\
& + \frac{1}{\lambda_i-\lambda_j} \frac{\partial \lambda_i}{\partial u_j} \sum_{l=1}^{g} (2l-1) \left(\frac{P_{l-1}^g(u_i)}{P_0^g(u_i)} - \frac{P_{l-1}^g(u_j)}{P_0^g(u_j)} \right) \sum_{k=l}^{g} q_k(\vec{u}) \tilde{\Gamma}_{k-l} \quad i \neq j,\ i,j = 1,\dots,2g+1.
\end{aligned}
\tag{4.26}
$$

Adding and subtracting the quantity $\dfrac{1}{\lambda_i-\lambda_j}\dfrac{\partial \lambda_i}{\partial u_j} w_j$ to (4.26), we can reduce it to the form

$$
\begin{aligned}
\partial_{u_j} w_i - \frac{1}{\lambda_i-\lambda_j} \frac{\partial \lambda_i}{\partial u_j}[w_i - w_j] = \partial_{u_j} q_g(\vec{u}) \Bigg(& \frac{2}{\lambda_i-\lambda_j} \frac{\partial \lambda_i}{\partial u_j} \frac{\prod_{l=1,l\neq j}^{2g+1}(u_j-u_l)}{P_0^g(u_j)} \\
& - \frac{\prod_{l=1,l\neq i}^{2g+1}(u_i-u_l)}{(u_i-u_j)P_0^g(u_i)} + \sum_{l=1}^{g} (2l-1) \frac{P_{l-1}^g(u_i)}{P_0^g(u_i)} \sum_{k=l}^{g} \tilde{\Gamma}_{k-l} u_j^{g-k} \Bigg)
\end{aligned}
\tag{4.27}
$$

The term in parenthesis in the right hand side of (4.27) does not depend on the initial data $f(u)$. It is identically zero for the analytic initial data (3.11) because in such case the w_i's satisfy (3.4) [8]. Therefore we can conclude that

$$
\partial_{u_j} w_i - \frac{1}{\lambda_i-\lambda_j} \frac{\partial \lambda_i}{\partial u_j}[w_i - w_j] = 0 \tag{4.28}
$$

for any smooth monotone increasing initial data $x = f(u)$.

b) *The $w_i(\vec{u})$'s satisfy the boundary conditions (3.7-3.10).*

In the following we use the sup-script g to denote the corresponding genus of the quantities we are referring to. We have the following relations. When $u_l = u_{l+1} = v$ for $l = 1, \ldots, 2g$, the $\tilde{\Gamma}_k$'s defined in (2.19) satisfy

$$\begin{aligned} \tilde{\Gamma}^g_k(u_1, \ldots, u_{l-1}, v, v, u_{l+2}, \ldots, u_{2g+1}) =& \tilde{\Gamma}^{g-1}_k(u_1, \ldots, u_{l-1}, u_{l+2}, \ldots, u_{2g+1}) \\ &- v\tilde{\Gamma}^{g-1}_{k-1}(u_1, \ldots, u_{l-1}, u_{l+2}, \ldots, u_{2g+1}), \quad k \geq 1,\ g > 1 \end{aligned} \tag{4.29}$$

and the $q_k(\vec{u})$'s defined in (4.19) satisfy

$$\begin{aligned} q^g_k(u_1, \ldots, u_{l-1}, v, v, u_{l+2}, \ldots, u_{2g+1}) - v q^g_{k+1}(u_1, \ldots, u_{l-1}, v, v, u_{l+2}, \ldots, u_{2g+1}) \\ = q^{g-1}_k(u_1, \ldots, u_{l-1}, u_{l+2}, \ldots, u_{2g+1}), \quad k = 1, \ldots, g-1,\ g > 1. \end{aligned} \tag{4.30}$$

For $u_i \neq u_l = u_{l+1} = v$ we have that

$$\begin{aligned} \partial_{u_i} q^{g-1}_{g-1}(u_1, \ldots, u_{l-1}, u_{l+2}, \ldots, u_{2g+1}) - \frac{1}{2} q^g_g(u_1, \ldots, u_{l-1}, v, v, u_{l+2}, \ldots, u_{2g+1}) = \\ (u_i - v)\partial_{u_i} q^g_g(u_1, \ldots, u_{l-1}, v, v, u_{l+2}, \ldots, u_{2g+1}), \end{aligned} \tag{4.31}$$

which follows from (4.20). When $u_l = u_{l+1} = v$ the polynomials $P^g_k(r)$'s defined in (2.7) satisfy the relation [16]

$$P^g_k(r) = (r - v)P^{g-1}_k(r), \quad k \geq 0\,. \tag{4.32}$$

Using the relations (4.29-4.32) we have that for $i \neq l,\ l+1,\ \ i = 1, \ldots, 2g+1$,

$$\begin{aligned} w^g_i(u_1, \ldots, u_{l-1}, v, v, u_{l+2}, \ldots, u_{2g+1}) =& 2\frac{\prod_{k=1, k\neq i,\, l,\, l+1}^{2g+1}(u_i - u_k)}{P^{g-1}_0(u_i)}\partial_{u_i} q^{g-1}_{g-1} \\ &+ \sum_{m=1}^{g-1}(2m-1)\frac{P^{g-1}_{m-1}(u_i)}{P^{g-1}_0(u_i)}\sum_{k=m}^{g-1} q^{g-1}_k \tilde{\Gamma}^{g-1}_{k-m} \\ &+ \left(\sum_{m=1}^{g}(2m-1)\frac{P^{g-1}_{m-1}(u_i)}{P^{g-1}_0(u_i)}\tilde{\Gamma}^{g-1}_{g-m} - \frac{\prod_{k=1, k\neq i,\, l,\, l+1}^{2g+1}(u_i - u_k)}{P^{g-1}_0(u_i)}\right) q^g_g, \end{aligned} \tag{4.33}$$

which reduces to the form

$$\begin{aligned} w^g_i(u_1, \ldots, u_{l-1}, v, v, u_{l+2}, \ldots, u_{2g+1}) =& w^{g-1}_i(u_1, \ldots, u_{l-1}, u_{l+2}, \ldots, u_{2g+1}) \\ &+ \left(\sum_{m=1}^{g}(2m-1)\frac{P^{g-1}_{m-1}(u_i)}{P^{g-1}_0(u_i)}\tilde{\Gamma}^{g-1}_{g-m} - \frac{\prod_{k=1, k\neq i,\, l,\, l+1}^{2g+1}(u_i - u_k)}{P^{g-1}_0(u_i)}\right) q^g_g. \end{aligned} \tag{4.34}$$

Using (2.21) the term in parenthesis in the right hand side of (4.34) turns out to be identically zero. Therefore the boundary conditions (3.8) are satisfied for any smooth monotone increasing initial data.

Since the functions $q_k(\vec{u})$'s in (4.19) and the $\tilde{\Gamma}_k(\vec{u})$'s defined in (2.19) are symmetric with respect to $u_1, u_2, \ldots, u_{2g+1}$ we immediately deduce from (4.32) that

$$w_l^g(u_1, \ldots, u_{l-1}, v, v, u_{l+2}, \ldots, u_{2g+1}) = w_{l+1}^g(u_1, \ldots, u_{l-1}, v, v, u_{l+2}, \ldots, u_{2g+1}) \tag{4.35}$$

so that the boundary conditions (3.7) are satisfied (for a more detailed analysis of this limit see section 5.1). When $g = 1$ we deduce from (4.35)

$$w_1(u_1, u_1, u_3) = w_2(u_1, u_1, u_3)$$

and from (4.31-4.32)

$$w_3(u_1, u_1, u_3) = 2(u_3 - u_1)\partial_{u_3} q_1(u_1, u_1, u_3) + q_1(u_1, u_1, u_3).$$

From (4.10) and (4.19) we get the relation

$$q_1(u_1, u_1, u_3) = f(u_3) + 2(u_1 - u_3)\partial_{u_3} q_1(u_1, u_1, u_3)$$

so that

$$w_3(u_1, u_1, u_3) = f(u_3).$$

An analogous result can be obtain when $u_2 = u_3$, so that the boundary conditions (3.9-3.10) are satisfied.

c) *Uniqueness.* We prove that when $f(u) \equiv 0$ then $w_i^g(\vec{u}) \equiv 0$ for all $b > u_1 > u_2 > \cdots > u_{2g+1} > a$, for $1 \leq i \leq 2g + 1$ and for any $g \geq 0$. The proof is by induction on g. For $g = 0$ the statement is satisfied.

For $g = 1$ we repeat the arguments of [6]. We fix u_2 and we consider the equation (3.4) with boundary condition (3.9-3.10), namely

$$\begin{aligned}
\frac{\partial w_1}{\partial u_3} &= a_{13}[w_1 - w_3] \\
\frac{\partial w_3}{\partial u_1} &= a_{31}[w_3 - w_1] \\
w_1(u_1, u_2, u_2) &= f(u_1) \equiv 0 \\
w_3(u_2, u_2, u_3) &= f(u_3) \equiv 0\,.
\end{aligned}$$

We can regard the above equations as a first-order linear ordinary differential equations with non homogeneous term. Integrating them we obtain a couple integral equation. By standard contraction mapping method it can be shown that when $f(u) \equiv 0$ this system has only zero solution, i.e. $w_1 = w_3 \equiv 0$ for (u_1, u_3) satisfying $b > u_1 > u_2 > u_3 > a$. Because of the arbitrariness of u_2, w_1 and w_3 vanish as a function of (u_1, u_2, u_3) and therefore, by (3.4) so does $w_2(\vec{u})$. Now we suppose the theorem

true for genus $g-1$ and we proof it for genus g. We fix $b > u_2 > u_3 > \cdots > u_{2g} > a$ and we consider the equation (3.4) for w_1^g and w_{2g+1}^g with boundary condition (3.7-3.8), namely

$$\begin{aligned}
\frac{\partial}{\partial u_{2g+1}} w_1^g &= a_{1(2g+1)}[w_1^g - w_{2g+1}^g] \\
\frac{\partial}{\partial u_1} w_{2g+1}^g &= a_{(2g+1)1}[w_{2g+1}^g - w_1^g] \\
w_1^g(u_1, u_2, \ldots, u_{2g}, u_{2g}) &= w_1^{g-1}(u_1, u_2, \ldots, u_{2g-1}, \hat{u}_{2g}, \hat{u}_{2g}) \equiv 0 \\
w_{2g+1}^g(u_2, u_2, \ldots, u_{2g}, u_{2g+1}) &= w_{2g+1}^{g-1}(\hat{u}_2, \hat{u}_2, u_3, \ldots, u_{2g}, u_{2g+1}) \equiv 0\,.
\end{aligned} \tag{4.36}$$

Repeating the arguments developed for genus $g = 1$ we may conclude that $w_1^g(\vec{u}) = w_{2g+1}^g(\vec{u}) \equiv 0$, for arbitrary $b > u_1 > u_2 > \cdots > u_{2g+1} > a$. We then repeat the above argument fixing $b > u_1 > u_3 > \cdots > u_{2g-1} > u_{2g+1} > a$ and considering the equations (3.4) for $w_2^g(\vec{u})$ and $w_{2g}^g(\vec{u})$, namely

$$\begin{aligned}
\frac{\partial}{\partial u_{2g}} w_2^g &= a_{2(2g)}[w_2^g - w_{2g}^g] \\
\frac{\partial}{\partial u_2} w_{2g}^g &= a_{(2g)2}[w_{2g}^g - w_2^g] \\
w_2^g(u_1, u_2, \ldots, u_{2g-1}, u_{2g+1}, u_{2g+1}) &= w_2^{g-1}(u_1, u_2, \ldots, u_{2g-1}, \hat{u}_{2g+1}, \hat{u}_{2g+1}) \equiv 0 \\
w_{2g}^g(u_1, u_1, u_3, \ldots, u_{2g}, u_{2g+1}) &= w_{2g}^{g-1}(\hat{u}_1, \hat{u}_1, u_3, \ldots, u_{2g}, u_{2g+1}) \equiv 0\,.
\end{aligned} \tag{4.37}$$

It can be easily shown that also $w_2^g(\vec{u}) = w_{2g}^g(\vec{u}) \equiv 0$ for arbitrary $b > u_1 > u_2 > \cdots > u_{2g+1} > a$. Repeating these arguments other $g-2$ times we conclude that $w_i^g(\vec{u}) \equiv 0$ for $1 \le i \le g,\ g+2 \le i \le 2g+1$ and for arbitrary $b > u_1 > u_2 > \cdots > u_{2g+1} > a$. Applying (3.4) and the boundary conditions (3.7)-(3.8) we can prove that also $w_{g+1}^g(\vec{u})$ is identically zero. The theorem is then proved. $\square$

In the next section we consider the problem of reality of the solution of the hodograph transform (3.6).

Remark 4.6 *The boundary conditions (3.7-3.10) guarantee the C^1-smoothness of the solution of the Whitham equations. Indeed it can be proved that the x derivatives $\partial_x u_i(x,t)$, $i = 1, \ldots 2g+1$, are continuous on the phase transition boundaries.*

5 An upper bound to the genus of the solution

In this section we give an upper bound to the genus of the solution of the Whitham equations for the monotone increasing smooth initial data $x = f(u)|_{t=0}$.

Theorem 5.1 (Second Main Theorem) *If the monotone increasing smooth initial data $f(u)$ satisfies the condition*

$$\frac{d^{2N+1}}{du^{2N+1}} f(u) > 0, \quad 1 \le N \in \mathbb{N}, \tag{5.1}$$

for all $u \in (a,b)$ except at one point, then the solution of the Whitham equations (1.1) has genus at most N.

Proof: the solution of the Whitham equations (1.1) for different $g \geq 0$ determines a decomposition of the $x-t$ plane, $t \geq 0$, into a number of domains D_g with g=0,1,2 ... , (see Fig. 5.3). To the inner part

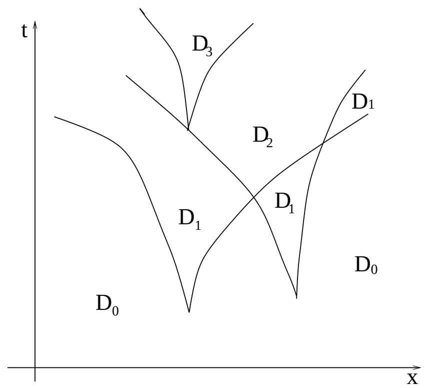

Figure 5.3: An example of decomposition of the $x-t$ plane

of each domain D_g it corresponds the g-phase solution $b > u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t) > a$ of the Whitham equations (1.1). The common boundaries of the domains D_g, $g \geq 0$, and D_{g+n}, $n \geq 1$, are the points of phase transition between the g-phase solution and the $(g+n)$-phase solution.

We will show that each domain D_g, $g \leq N$, does not have a common boundary with any of the domains D_m, $m > N$. Since the set of domains $\{D_g\}_{g\leq N}$ is not empty because $D_0 \neq \emptyset$ the set $\{D_g\}_{g>N}$ must be empty. Indeed on the contrary the $x-t$ plane, $t \geq 0$, which is a connected set, would be split into a number of domains whose union forms a disconnected set.
Before determining the boundaries of the domains D_g, $g \geq 0$, we need to study more in detail the hodograph transform (3.6).

Proposition 5.2 *Let us consider the polynomial*

$$Z^g(r) := -xP_0^g(r) - 12tP_1^g(r) + R^g(r), \tag{5.2}$$

where $R^0(r) = f(u)$ *and* $R^g(r)$, $g > 0$, *is given by the expression*

$$R^g(r) = 2\sum_{k=1}^{2g+1} \partial_{u_k} q_g(\vec{u}) \prod_{l=1, l\neq k}^{2g+1} (r - u_l) + \sum_{k=1}^{g} q_k(\vec{u}) \sum_{l=1}^{k} (2l-1)\tilde{\Gamma}_{k-l} P_{l-1}^g(r), \tag{5.3}$$

with the polynomials $P_l^g(r)$, $l \geq 0$, *defined in (2.7) and the functions* $q_k(\vec{u})$, $k = 1, \ldots, g$, *defined in (4.10). Then the hodograph transform (3.6) is equivalent, for* $g > 0$, *to the equation*

$$Z^g(r) \equiv 0, \quad g > 0. \tag{5.4}$$

Proof: We observe that the $w_i(\vec{u})$'s defined in (4.11) are given by the ratio $w_i(\vec{u}) = \dfrac{R^g(u_i)}{P_0^g(u_i)}$, $i = 1,\dots,2g+1$, where $R^g(r)$ is the polynomial defined in (5.3). Hence we can write the hodograph transform (3.6) in the form

$$[-xP_0^g(r) - 12tP_1^g(r) + R^g(r)]_{r=u_i} = 0, \quad i = 1,\dots,2g+1. \tag{5.5}$$

For $g > 0$, $Z^g(r)$ is a polynomial of degree $2g$ and because of (5.5) it must have at least $2g+1$ real zeros. Therefore it is identically zero. Hence for $g > 0$, (5.4) is equivalent to (5.5) and (3.6). $\square$

In the following analysis we give some conditions for the existence of a real solution $b > u_1(x,t) > u_2(x,t) > \dots, u_{2g+1}(x,t) > a$ of the hodograph transform (3.6). For the purpose let us consider the function $\Psi^g(r;\vec{u})$ which solves the boundary value problem

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial u_i}\Psi^g(r;\vec{u}) - \dfrac{\partial}{\partial u_j}\Psi^g(r;\vec{u}) = 2(u_i - u_j)\dfrac{\partial^2}{\partial u_i \partial u_j}\Psi^g(r;\vec{u}), \quad i \neq j, \;\; i,j = 1,\dots 2g+1 \\ \dfrac{\partial}{\partial r}\Psi^g(r;\vec{u}) - 2\dfrac{\partial}{\partial u_j}\Psi^g(r;\vec{u}) = 2(r - u_j)\dfrac{\partial^2}{\partial r \partial u_j}\Psi^g(r;\vec{u}), \quad j = 1,\dots 2g+1 \\ \Psi^g(r;\underbrace{r,\dots,r}_{2g+1}) = \dfrac{2^g}{(2g+1)!!}f^{(g)}(r) \end{array}\right. \tag{5.6}$$

where $f^{(g)}(r)$ is the gth derivative of the smooth monotone increasing initial data $f(u)$. From the results of Sec. 4 we are able to integrate (5.6) obtaining

$$\begin{aligned}\Psi^g(r;\vec{u}) = & \frac{1}{K}\int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1} d\xi_1 d\xi_2 \dots d\xi_{2g+1}(1+\xi_{2g+1})^g(1+\xi_{2g})^{g-\frac{1}{2}}\dots(1+\xi_2)^{\frac{1}{2}} \times \\ & \frac{f^{(g)}(\frac{1+\xi_{2g+1}}{2}(\dots(\frac{1+\xi_2}{2}(\frac{1+\xi_1}{2}r + \frac{1-\xi_1}{2}u_1) + \frac{1-\xi_2}{2}u_2) + \dots) + \frac{1-\xi_{2g+1}}{2}u_{2g+1})}{\sqrt{(1-\xi_1)(1-\xi_2)\dots(1-\xi_{2g+1})}}\end{aligned} \tag{5.7}$$

where $K = \dfrac{2^g}{(2g+1)!!}\prod_{j=2}^{2g+2} C_j$ and the C_j's have been defined in (4.14). The function $\Psi^g(r;\vec{u})$ is symmetric with respect to the variables $b > u_1 > u_2 > \dots > u_{2g+1} > a$. For the initial data (3.11), the expression of $\Psi^g(r;\vec{u})$ in (5.7) is equivalent to (4.2).

Proposition 5.3 *Let us consider the function*

$$\Phi^g(r;\vec{u}) = \partial_r\Psi^g(r;\vec{u}) + \sum_{i=1}^{2g+1}\partial_{u_i}\Psi^g(r;\vec{u}), \tag{5.8}$$

where $\Psi^g(r;\vec{u})$ has been defined in (5.7). If $u_1(x,t) > u_2(x,t) > \dots > u_{2g+1}(x,t)$ satisfy the Whitham equations (1.1) then the function $\Phi^g(r;\vec{u})$ has, in the r variable, at least one real zero in each of the intervals (u_{2k}, u_{2k-1}), $k = 1,\dots,g$, $g > 0$.

Proof: If the functions $u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t)$ satisfy the Whitham equations then by proposition 5.2 the polynomial $Z^g(r) \equiv 0$. Therefore

$$
\begin{aligned}
0 \equiv & \int_{\alpha_k} \frac{Z^g(r)}{\mu(r)} dr = \int_{\alpha_k} \frac{-xP_0^g(r) - 12tP_1^g(r) + R^g(r)}{\mu(r)} dr \\
= & \int_{\alpha_k} 2 \frac{\sum_{i=1}^{2g+1} \partial_{u_i} q_g(\vec{u}) \prod_{j=1, j\neq i}^{2g+1}(r-u_j)}{\mu(r)} dr, \quad k = 1, \dots, g.
\end{aligned}
\tag{5.9}
$$

In the third equality of (5.9) we have used the fact that

$$
\int_{\alpha_k} \frac{P_l^g(r)}{\mu(r)} dr = \int_{\alpha_k} \sigma_l^g(r) = 0, \quad l \geq 0, \; k = 1, \dots, g,
$$

because of the normalization conditions (2.5). The function $\Psi^g(r;\vec{u})$ satisfies the relations

$$
\frac{\Psi^g(r;\vec{u})}{r-u_i} - \frac{\Psi^g(u_i;\vec{u})}{r-u_i} = 2\partial_{u_i}\Psi^g(r;\vec{u}), \quad 2\partial_{u_i} q_g(\vec{u}) = \Psi^g(u_i;\vec{u}) \tag{5.10}
$$

which can be easily obtained from (5.7). Using (5.10) we can rewrite the last term in (5.9) in the form

$$
\begin{aligned}
0 = & \int_{\alpha_k} 2 \frac{\sum_{i=1}^{2g+1} \partial_{u_i} q_g(\vec{u}) \prod_{j=1, j\neq i}^{2g+1}(r-u_j)}{\mu(r)} dr, \quad k = 1, \dots, g, \\
=& 2 \int_{u_{2k}}^{u_{2k-1}} \mu(r) \left(\sum_{i=1}^{g} \frac{\Psi^g(r;\vec{u})}{r-u_i} - 2\partial_{u_i}\Psi^g(r;\vec{u}) \right) dr \\
=& -4 \int_{u_{2k}}^{u_{2k-1}} \mu(r) \left(\partial_r \Psi^g(r;\vec{u}) + \sum_{i=1}^{g} \partial_{u_i}\Psi^g(r;\vec{u}) \right) dr, \quad k = 1, \dots, g,
\end{aligned}
$$

where the last equality has been obtained integrating by parts. Using the definition of $\Phi^g(r;\vec{u})$ in (5.8) we rewrite the above relation in the form

$$
0 = -4 \int_{u_{2k}}^{u_{2k-1}} \mu(r) \Phi^g(r;\vec{u}) dr, \quad k = 1, \dots, g. \tag{5.11}
$$

Relation (5.11) is satisfied only if the function $\Phi^g(r;\vec{u})$ changes sign at least once in each of the intervals (u_{2k}, u_{2k-1}), $k = 1, \dots, g$. $\square$

In the following we are going to determine the equations which describe, on the $x - t \geq 0$ plane, the boundary between the domains D_g and D_{g+1}.

The boundary between the domains D_g and D_{g+1} represents a singular behavior in the solution of the g or $(g+1)$-phase equations. As we have shown in the example on fig 1.1, the boundary between the domains D_0 and D_1 is described by the curves $x^{\pm}(t)$ where $u_2(x,t) = u_3(x,t)$ and $u_1(x,t) = u_2(x,t)$ respectively and by the point x_0, t_0 of gradient catastrophe of the zero phase solution.
In a similar way the generic boundary between the domains D_g and D_{g+1} is described by
a) the curves $x_g^{\pm}(t)$ where two Riemann invariants of the $(g+1)$-phase solution coalesce;

b) the point of gradient catastrophe x_c, $t_c > 0$ of the g-phase solution, namely the point where one of the $2g+1$ Riemann invariants has a vertical inflection point. The equations determining the point of gradient catastrophe of the g-phase solution can also be obtained considering the limit of the $g+1$ phase solution when three Riemann invariants coalesce.

To treat case a) we consider the Riemann surface $\mathcal{S}_{g+1}$ of genus $g+1$ given by the equations

$$\tilde{\mu}^2 = (r - v - \sqrt{\delta})(r - v + \sqrt{\delta})\mu^2, \quad v \in \mathbb{R},$$

$$\mu^2 = \prod_{j=1}^{2g+1}(r - u_j), \quad b > u_1 > u_2 > \cdots > u_{2g+1} > a,$$

where $v \neq u_j$, $j = 1, \ldots 2g+1$, and $0 < \delta \ll 1$. The Riemann invariants are the $2g+3$ variables $\tilde{u}_1 = v + \sqrt{\delta}$, $\tilde{u}_2 = v - \sqrt{\delta}$, $u_1 > u_2 > \cdots > u_{2g+1}$. We suppose $\tilde{u}_1, \tilde{u}_2 \neq u_j$, $j = 1, \ldots, 2g+1$. The hodograph transform (3.6) for these $2g+3$ variables has two different behavior for $\delta \to 0$ when v belongs to one of the bands

$$(u_{2g+1}, u_{2g}) \cup (u_{2g-1}, u_{2g-2}) \cup \cdots \cup (u_3, u_2) \cup (u_1, b) \tag{5.12}$$

or gaps

$$(a, u_{2g+1}) \cup (u_{2g}, u_{2g-1}) \cup \cdots \cup (u_4, u_3) \cup (u_2, u_1). \tag{5.13}$$

We call *leading edge* of the phase transition boundary the case in which v belongs to the bands. We call *trailing edge* of the phase transition boundary the case in which v belongs to the gaps (5.13).

Theorem 5.4 *The leading edge of the phase transition boundary between the g-phase solution and the $(g+1)$-phase solution is described by the system*

$$\begin{cases} \Phi^g(v; \vec{u}) - 6t\,\epsilon_{g0} = 0 \\ \partial_v \Phi^g(v; \vec{u}) = 0 \\ x = \left[-12t\dfrac{P_1^g(r)}{P_0^g(r)} + \dfrac{R^g(r)}{P_0^g(r)}\right]_{r=u_i} \qquad , \quad i = 1, \ldots, 2g+1, \; g \geq 0 \end{cases} \tag{5.14}$$

where $v \in (u_{2j+1}, u_{2j})$, $0 \leq j \leq g$, $u_0 = +b$, the function $\epsilon_{g0} = 1$ for $g = 0$ and zero otherwise, the function $\Phi^g(r; \vec{u})$ has been defined in (5.8) and the polynomial $R^g(r)$ has been defined in (5.3). We assume $(\partial_v)^2 \Phi^g(v; \vec{u}) \neq 0$ and $\Phi^g(u_i; \vec{u}) \neq 0$, $i = 1, \ldots, 2g+1$, on the solution of (5.14).

Remark 5.5 *System (5.14) is a system of $2g+3$ equations in $2g+4$ unknowns x, t, v and $u_1 > u_2 > \cdots > u_{2g+1}$. If system (5.14) is uniquely solvable for real x, v and $u_1 > u_2 > \cdots > u_{2g+1}$ as a function of $t \geq 0$, then a phase transition between the g-phase solution and the $(g+1)$-phase solution occurs. The curve $x_g^- = x_g^-(t)$ describes on the $x - t \geq 0$ plane the boundary between the domains D_g and D_{g+1}*

associated to the leading edge. The conditions $(\partial_v)^2\Phi^g(v;\vec{u}) \neq 0$ *and* $\Phi^g(u_i;\vec{u}) \neq 0$, $i = 1, \ldots, 2g+1$, *on the solution of (5.14) exclude higher order degeneracy in the transition. Indeed in such case it can be proved that in a neighborhood of the solution* $x(t)$, $v(t)$, $u_1(t) > u_2(t) > \cdots > u_{2g+1}(t)$ *of (5.14) the* $(g+1)$*-phase solution is uniquely defined.*

Theorem 5.6 *The trailing edge of the phase transition boundary between the* g*-phase solution and the* $(g+1)$*-phase solution is described by the solution of the system*

$$\begin{cases} \Phi^g(v;\vec{u}) - 6t\,\epsilon_{g0} = 0 \\ \displaystyle\int_v^{u_{2j-1}} (\Phi^g(r;\vec{u}) - 6t\,\epsilon_{g0})\mu(r)dr = 0 \\ x = \left[-t\dfrac{P_1^g(r)}{P_0^g(r)} + \dfrac{R^g(r)}{P_0^g(r)}\right]_{r=u_i}, \quad i = 1, \ldots, 2g+1,\ g > 0 \end{cases} \tag{5.15}$$

where $v \in (u_{2j}, u_{2j-1})$, $1 \leq j \leq g+1$, $u_{2g+2} = a$ *and the function* $\Phi^g(r;\vec{u})$ *has been defined in (5.8). We assume* $\partial_v\Phi^g(v;\vec{u}) \neq 0$ *and* $\Phi^g(u_i;\vec{u}) \neq 0$, $i = 1, \ldots, 2g+1$, *on the solution of (5.15).*

If system (5.15) is uniquely solvable for real x, v and $u_1 > u_2 > \cdots > u_{2g+1}$ as a function of $t \geq 0$, then a phase transition between the g-phase solution and the $(g+1)$-phase solution occurs. The curve $x_g^+ = x_g^+(t)$ describes on the $x - t \geq 0$ plane the boundary between the domains D_g and D_{g+1} associated to the trailing edge.
The following theorem enables one to determine a point of gradient catastrophe of the g-phase solution. This point can be obtained either as a limit of the $(g+1)$ phase solution when three Riemann invariants coalesce or imposing a vertical inflection point on the g-phase solution.

Theorem 5.7 *Let us consider the system*

$$\begin{cases} \partial_r\Phi^g(r;\vec{u})|_{r=u_l} = 0 \\ \Phi^g(u_l;\vec{u}) - 6t\,\epsilon_{g0} = 0 \\ x = \left[-12t\dfrac{P_1^g(r)}{P_0^g(r)} + \dfrac{R^g(r)}{P_0^g(r)}\right]_{r=u_i}, \quad i = 1, \ldots, 2g+1,\ g \geq 0 \end{cases} \tag{5.16}$$

where $\epsilon_{g0} = 1$ *for* $g = 0$ *and zero otherwise, the function* $\Phi^g(r;\vec{u})$ *has been defined in (5.8). When system (5.16) is uniquely solvable for* x_c, $t_c \geq 0$ *and* $b > u_1(x_c,t_c) > u_2(x_c,t_c) > \cdots > u_{2g+1}(x_c,t_c) > a$, *then the* g*-phase solution has a point of gradient catastrophe on the* u_l *branch,* $1 \leq l \leq 2g+1$.

To avoid higher order degeneracies in the transition we impose the condition

$$(\partial_r)^2\Phi^g(r;\vec{u}(x_c,t_c))|_{r=u_l(x_c,t_c)} \neq 0. \tag{5.17}$$

Indeed we can prove that the above condition guarantees that the genus of solution of the Whitham equations increases at most by one in the neighborhood of the point of gradient catastrophe. Therefore it is legitimate to consider the point of gradient catastrophe that solve (5.16) and satisfies (5.17) as a point of the boundary between the domains D_g and D_{g+1}. The condition (5.17) in not essential in the genus $g = 0$ case as illustrated in Theorem 3.2.

Remark 5.8 *We observe that both systems (5.14) and (5.15) in the limit $v \to u_l$, $l = 1, \dots, 2g+1$, coincide with system (5.16).*

Theorems 5.4, 5.6 and 5.7 characterize all types of boundaries between the g-phase solution and the $(g+1)$-phase solution, namely the leading edge, the trailing edge and the points of gradient catastrophe. We will prove these theorems in the next section.

Example 3.1 For $x = u^k$, $k = 3, 5, 7, \dots$, the solution of the Whitham equations has genus at most equal to one [12],[6]. On the $x - t$, $t \geq 0$, plane we have only the domains D_0 and D_1. The one phase solution is defined within the cusp $x_-(k)\, t^{\frac{k}{k-1}} < x < x_+(k)\, t^{\frac{k}{k-1}}$, where $x_-(k) < x_+(k)$ are two real constants and $t > 0$. The point $x = 0, t = 0$ is the point of gradient catastrophe of the zero-phase solution. The constants $x_-(k)$ and $x_+(k)$ can be computed explicitly. From (5.7) and (5.8) we obtain

$$\Phi_0(r, u) = \frac{1}{2\sqrt{r-u}} \int_u^r \frac{f'(\xi) d\xi}{\sqrt{r-\xi}}. \tag{5.18}$$

On the leading edge where $u_1 = u_2 = v$ and $u_3 = u$, $v > u$, (5.14) is equivalent to the equations

$$\int_u^v \frac{(\xi - u) f''(\xi)}{\sqrt{v-\xi}} d\xi = 0, \qquad \int_u^v \frac{f'(\xi) - 6t}{\sqrt{v-\xi}} d\xi = 0, \quad x = -6tu + f(u). \tag{5.19}$$

On the trailing edge where $u_1 = u$, $u_2 = u_3 = v$, $u > v$, system (5.15) is equivalent, for $g = 0$ to

$$\int_v^u (f'(\xi) - 6t)\sqrt{v-\xi}\, d\xi = 0, \qquad \int_u^v \frac{f'(\xi) - 6t}{\sqrt{v-\xi}} d\xi = 0, \quad x = -6tu + f(u). \tag{5.20}$$

Equations (5.19) and (5.20) have already been obtained in [6]. Solving system (5.19) for t when $f(u) = u^k$, $k = 3, 5, 7, \dots$ we obtain $x_-(k)$ [13],[17]

$$x_-(k) = -6\, \frac{k-1}{k} (2z_-(k) - 1) \left[\frac{6}{k} (1 + 2(k-1) z_-(k)) \right]^{\frac{1}{k-1}}, \quad k = 3, 5, 7, \dots, \tag{5.21}$$

where $z_-(k) > 1$ is the unique real solution of $F(-k+2, 2, \frac{5}{2}; z) = 0$. Here $F(a, b, c; z)$ is the hypergeometric series. The quantity $x_+(k)$ is obtained from (5.20)

$$x_+(k) = 2\, \frac{k-1}{k} (2z_+(k) - 3) \left[\frac{2}{k} (3 + 2(k-1) z_+(k)) \right]^{\frac{1}{k-1}}, \quad k = 3, 5, 7, \dots, \tag{5.22}$$

where the number $z_+(k) > 1$ is the unique real solution of the equation $F(-k+2, 2, \frac{7}{2}; z) = 0$. We give some numerical values: $x_-(3) = -12\sqrt{3},\ x_-(5) = -16.85,\ x_-(7) = -16.21,\ x_-(9) = -16.09,$ $x_+(3) = \frac{4}{3}\sqrt{\frac{5}{3}},\ x_+(5) = 1.58,\ x_+(7) = 1.61,\ x_+(9) = 1.72.$

Theorems 5.4, 5.6 and 5.7 can be generalized to the case of multiple phase transitions.

Theorem 5.9 *The transition boundary between the domains D_g, $g \geq 0$ and D_{g+n}, $n > 1$, having n_1 leading edges and n_2 trailing edges and n_3 points of gradient catastrophes, $n_1 + n_2 + n_3 = n$, $n_1, n_2, n_3 \geq 0$, is described by the solution of the system*

$$\begin{cases} \partial_{v_k}\Phi^g(v_k;\vec{u}) = 0, \quad k = 1,\dots,n_1 \\ \Phi^g(v_k;\vec{u}) - 6t\,\epsilon_{g0} = 0, \quad k = 1,\dots,n_1 \\ \Phi^g(y_l;\vec{u}) - 6t\,\epsilon_{g0} = 0, \quad l = 1,\dots,n_2 \\ \int_{y_l}^{u_{2j_l-1}}(\Phi^g(r;\vec{u}) - 6t\,\epsilon_{g0})\mu(r)dr = 0, \quad l = 1,\dots,n_2 \\ \partial_r\Phi^g(r;\vec{u})|_{r=u_{j_m}} = 0, \quad m = 1,\dots,n_3 \\ \Phi^g(u_{j_m};\vec{u}) - 6t\,\epsilon_{g0} = 0 \quad m = 1,\dots,n_3 \\ \\ x = \left[-12t\dfrac{P_1^g(r)}{P_0^g(r)} + \dfrac{R^g(r)}{P_0^g(r)}\right]_{r=u_i}, \quad i = 1,\dots,2g+1, \end{cases} \tag{5.23}$$

where $v_k \in (u_{2j_k}, u_{2j_k+1})$, $0 \leq j_k \leq g$, $k = 1,\dots,n_1$ and $y_l \in (u_{2j_l-1}, u_{2j_l})$, $1 \leq j_l \leq g+1$, $l = 1,\dots,n_2$.

Remark 5.10 *The transition between the g-phase solution and the $(g+n)-$phase solution, $n > 1$, is highly non generic (cfr. Fig 5.23). Indeed system (5.23) is a systems of $2n+2g+1$ equations in n_1+n_2+2g+3 unknowns $v_1,\dots,v_{n_1}$, $y_1,\dots,y_{n_2}$, $u_1,\dots,u_{2g+1}$, x and $t \geq 0$. Therefore if system (5.23) admits a real solution, $n+n_3-2$ variables are functions of all the others.*
Theorem 5.9 includes all the degenerate cases that have been excluded in Theorems 5.4, 5.6 and 5.7. For example let us consider a transition with a double-leading edge described by the variables v_1, v_2 and $u_1 > u_2 > \cdots > u_{2g+1}$ which solve (5.23) with $n_1 = 2$, $n_2 = 0$, $n_3 = 0$. In the limit $v_1 \to v_2 = v$ we get a single degenerate leading edge. The variables v, $u_1 > u_2 > \cdots > u_{2g+1}$ satisfy (5.4) and the equations $(\partial_v)^2\Phi^g(v;\vec{u}) = 0$ and $(\partial_v)^3\Phi^g(v;\vec{u}) = 0$. Therefore we regard such degenerate single leading edge as a point of the boundary of the domains D_g and D_{g+2}.

Proposition 5.11 *If the domains D_g and D_{g+1} have a common boundary, then the function*

$$\Phi^g(r;\vec{u}) - 6t\epsilon_{g0}$$

has at least $g+2$ real zeros in the r variable for $u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t)$ satisfying (5.4) and for $t > 0$.

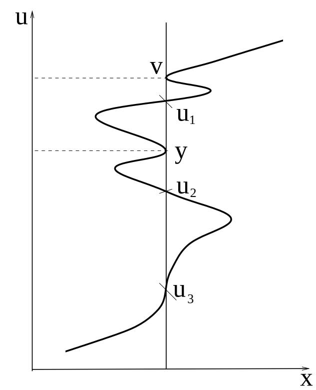

Figure 5.4: an example of phase transition between the one-phase solution and the 4-phase solution having one leading edge, one trailing edge and one point of gradient catastrophe at u_3.

Proof: The domains D_g and the D_{g+1} have a common boundary if one of the systems (5.14), (5.15) or (5.16) is solvable for some $t > 0$. We first consider the leading edge. For $g = 0$ the statement is obvious from system (5.14). For $g > 0$ system (5.14) can have a solution if the function $\Phi^g(r; \vec{u})$ has a double zero at $r = v$ when v belongs to the bands (5.12). Combining this observation with proposition 5.3 we immediately obtain the statement. As regarding the trailing edge, the theorem is obvious for $g = 0$. For $g > 0$ and $v \in (a, u_{2g+1})$ system (5.15) can be satisfied if the function $\Phi^g(r; \vec{u})$ has at least two zeros in the interval (a, u_{2g+1}). Combining this observation with proposition 5.3 we obtain the statement. When $v \in (u_{2j}, u_{2j-1})$, $1 \le j \le g$, (5.11) and (5.15) are satisfied if the function $\Phi^g(r; \vec{u})$ has at least three real zeros in the interval (u_{2j}, u_{2j-1}), $1 \le j \le g$, and changes sign at least once in each of the intervals (u_{2k}, u_{2k-1}), $k = 1, \ldots, g$, and $k \neq j$. Therefore $\Phi^g(r; \vec{u})$ has at least $g + 2$ real zeros at the trailing edge. If the point of the boundary between the domains D_g and the D_{g+1} corresponds to a point of gradient catastrophe of the g-phase solution the statement is obvious from system (5.16) and proposition 5.3. Therefore, on the phase transition boundary between the domains D_g and D_{g+1} the function $\Phi^g(r; \vec{u})$ has at least $g + 2$ real zeros in the r variable, $b > r > a$, for $b > u_1(x, t) > u_2(x, t) > \cdots > u_{2g+1}(x, t) > a$ and for $t > 0$. . $\square$

Proposition 5.11 can be generalized to multiple phase transitions.

Proposition 5.12 *On the phase transition boundary between the domains D_g and the D_{g+n} the function*

$$\Phi^g(r; \vec{u}) - 6\, t\epsilon_{g0}$$

has at least $g + 2n$ real zeros in the r variable for $b > u_1(x, t) > u_2(x, t) > \cdots > u_{2g+1}(x, t) > a$ and for $t > 0$.

The proof is analogous to that of proposition 5.11.

Lemma 5.13 *If the smooth initial data $x = f(u)|_{t=0}$ satisfies (5.1), then the function*

$$\Phi^g(r;\vec{u})$$

has at most $2N-g$ real zeros (counting multiplicity) in the r variable for $0 < g \leq 2N$ and for any real x, $t \geq 0$ and $b > u_1 > u_2 > \cdots > u_{2g+1} > a$.

Proof: for proving the lemma we need the following elementary result. If the real smooth function $\xi(r)$ satisfies the condition

$$\frac{d^m}{dr^m}\xi(r) > 0,\ 0 \leq m \in \mathbb{N},$$

for all r belonging to the domain of ξ, then $\xi(r)$ has at most m real zero.
Using (5.1), (5.6) and (5.8) we obtain

$$\frac{\partial^{2N-g}}{\partial r^{2N-g}}\Phi^g(r;\vec{u}) = \int_{-1}^{1}\cdots\int_{-1}^{1} d\xi_1\dots d\xi_{2g+1}(1+\xi_{2g+1})^{2N}(1+\xi_{2g})^{2N-\frac{1}{2}}\dots(1+\xi_2)^{2N-g+\frac{1}{2}}(1+\xi_1)^{2N-g}$$

$$\times\frac{f^{(2N+1)}(\frac{1+\xi_{2g+1}}{2}(\dots(\frac{1+\xi_2}{2}(\frac{1+\xi_1}{2}r+\frac{1-\xi_1}{2}u_1)+\frac{1-\xi_2}{2}u_2)+\dots)+\frac{1-\xi_{2g+1}}{2}u_{2g+1})}{2^{(2g+1)(2N-g)}\,K\,\sqrt{(1-\xi_2)(1-\xi_2)\dots(1-\xi_{2g+1})}} > 0 \tag{5.24}$$

for $g > 0$, for any real r and for any fixed real $u_1 > u_2 > \cdots > u_{2g+1}$ belonging to the interval (a,b). Therefore $\Phi^g(r;\vec{u})$ has at most $2N-g$ real zeros in the r variable for $0 < g \leq 2N$ and for any fixed real $b > u_1 > u_2 > \cdots > u_{2g+1} > a$. $\square$

Lemma 5.14 *If the smooth monotone increasing initial data $x = f(u)|_{t=0}$ satisfies (5.1), then the function*

$$\Phi^0(r;u) - 6t$$

has at most $2N$ real zeros (counting multiplicity) in the r variable, for any $t \geq 0$ and $u \neq u^*$, where u^* is the unique solution of the equation $f^{(2N+1)}(u^*) = 0$. For what $\Phi^0(r;u^*) - 6t$ is concerned we have two possibilities: either it has at most $2N$ real zeros (counting multiplicity) in the r variable for any $t \geq 0$ and some of these zeros are distinct, or it has at most two real zeros in the r variable for any $t > 0$ except for $t = t^* \geq 0$ where $r = u^*$ is a zero of multiplicity higher than $2N$.

Proof: Since $f'(u) \geq 0$ and because of (5.1) we have two possibilities: either $f'(u)$ has at most N simple minima and some of these minima are distinct, or $f'(u)$ is a non negative convex function with a minimum at $u = u^*$, where u^* is the unique solution of the equation $f^{(2N+1)}(u^*) = 0$. From the above considerations the lemma easily follows. $\square$

We remark that when $f'(u)$ is a convex function, we can apply Tian's Theorem 3.2. Indeed in such case we have $f'''(u) > 0$ for all $u \neq u^*$ and therefore the solution of the Whitham equations has genus at most one.

We continue the proof of the second main theorem. In the following we exclude the case in which $f'(u)$ is a convex function. From proposition 5.12 we deduce that when the domains D_g and D_m, $m > g \geq 0$, have a common boundary the function $\Phi^g(r;\vec{u}) - 6t\epsilon_{g0}$ defined in (5.8) has at least $2m-g$ real zeros for some values of x and t. From lemma 5.13 the function $\Phi^g(r;\vec{u}) - 6t\epsilon_{g0}$ defined in (5.8) has at most $2N-g$ real zeros for $g \leq 2N$ when the initial data $f(u)$ satisfies (5.1). Therefore

$$2N - g \geq 2m - g \quad \text{or} \quad m \leq N. \tag{5.25}$$

This shows that the set of domains $\{D_g\}_{0\leq g\leq N}$ does not have common boundaries with any of the domains in the set $\{D_m\}_{m>N}$. The second main theorem is proved. □

5.1 Phase transitions

In this subsection we prove theorems 5.4, 5.6, 5.7 and (5.9).
In the following we denote with a sup-script the genus of the quantity we are referring to. Whenever we omit the sup-script we are referring to genus g quantity.

We denote with $\sigma_k^{g+1} = \sigma_k^{g+1}(r,\delta,v)$ the normalized Abelian differential of the second kind with pole at infinity of order $2k+2$ defined on the surface $\mathcal{S}_{g+1}$ of genus $g+1$

$$\tilde{\mu}^2 = (r - v - \sqrt{\delta})(r - v + \sqrt{\delta})\mu^2, \quad v \in \mathbb{R},$$

$$\mu^2 = \prod_{j=1}^{2g+1}(r - u_j), \quad u_1 > u_2 > \cdots > u_{2g+1},$$

where $v \neq u_j$, $j = 1, \ldots 2g+1$, and $0 < \delta \ll 1$. We define the polynomials

$$P_k^{g+1}(r,\delta,v) = \tilde{\mu}(r)\frac{\sigma_k^{g+1}(r,\delta,v)}{dr}.$$

The expression of the polynomial in (5.3) for the Riemann invariants $\tilde{u}_1 = v + \sqrt{\delta}$, $\tilde{u}_2 = v - \sqrt{\delta}$ and $u_1 > u_2 > \cdots > u_{2g+1}$ reads

$$\begin{aligned} R^{g+1}(r,\delta,v) = {} & 2\sum_{i=1}^{2g+1}\partial_{u_i}q_{g+1}^{g+1}\prod_{j=1,j\neq i}^{2g+1}(r-u_j)(r-\tilde{u}_1)(r-\tilde{u}_2) + 2\mu^2(r)(r-\tilde{u}_1)\partial_{\tilde{u}_2}q_{g+1}^{g+1} \\ & +2\mu^2(r)(r-\tilde{u}_2)\partial_{\tilde{u}_1}q_{g+1}^{g+1} + \sum_{k=1}^{g+1}q_k^{g+1}\sum_{l=1}^{k}(2l-1)\tilde{\Gamma}_{k-l}^{g+1}P_{l-1}^{g+1}(r,\vec{u},\delta) \end{aligned} \tag{5.26}$$

where $q_k^{g+1} = q_k^{g+1}(\tilde{u}_1, \tilde{u}_2, u_1, \dots, u_{2g+1})$, $k = 1, \dots, g+1$ and $\tilde{\Gamma}_k^{g+1} = \tilde{\Gamma}_k^{g+1}(\tilde{u}_1, \tilde{u}_2, u_1, \dots, u_{2g+1})$, $k \geq 0$. We will sometimes omit the explicit dependence of $\sigma_k^{g+1}(r, \delta, v)$, $P_k^{g+1}(r, \delta, v)$ and $R^{g+1}(r, v, \delta)$ on the parameters v and δ.

Proof of Theorem 5.4.

We write the $(g+1)$-phase solution (3.6) for the variables $\tilde{u}_1 = v + \sqrt{\delta}$, $\tilde{u}_2 = v - \sqrt{\delta}$, $u_1 > u_2 > \dots > u_{2g+1}$ in the form

$$\begin{cases} 0 = \dfrac{1}{\tilde{u}_1 - \tilde{u}_2} \left(\left[-12t \dfrac{P_1^{g+1}(\tilde{u}_1, \delta)}{P_0^{g+1}(\tilde{u}_1, \delta)} + \dfrac{R^{g+1}(\tilde{u}_1, \delta)}{P_0^{g+1}(\tilde{u}_1, \delta)} \right] - \left[-12t \dfrac{P_1^{g+1}(\tilde{u}_2, \delta)}{P_0^{g+1}(\tilde{u}_2, \delta)} + \dfrac{R^{g+1}(\tilde{u}_2, \delta)}{P_0^{g+1}(\tilde{u}_2, \delta)} \right] \right), \\[2ex] x = \left[-12t \dfrac{P_1^{g+1}(r, \delta)}{P_0^{g+1}(r, \delta)} + \dfrac{R^{g+1}(r, \delta)}{P_0^{g+1}(r, \delta)} \right]_{r = \tilde{u}_2}, \\[2ex] x = \left[-12t \dfrac{P_1^{g+1}(r, \delta)}{P_0^{g+1}(r, \delta)} + \dfrac{R^{g+1}(r, \delta)}{P_0^{g+1}(r, \delta)} \right]_{r = u_i}, \quad i = 1, \dots, 2g+1 \end{cases} \tag{5.27}$$

The reason to write the hodograph transform (3.6) in the form (5.27) is that system (5.27) is non degenerate at the phase transition $\tilde{u}_1 = \tilde{u}_2$.

We show that system (5.27) reduces to system (5.14) when $\tilde{u}_1 = \tilde{u}_2$ or $\delta = 0$.

It is clear from Theorem 4.2 and proposition 5.2 that the last $2g+1$ equations in (5.27) in the limit $\tilde{u}_1 = \tilde{u}_2$ reduce to those of system (5.14).

For computing the limit $\tilde{u}_1 \to \tilde{u}_2$ of the first two equations in (5.27) we first need to study the behavior of the differentials $\sigma_k^{g+1}(r, \delta, v)$, $k \geq 0$, when $\delta \to 0$. When v belongs to one of the bands (5.12), the expansion of $\sigma_k^{g+1}(r, \delta, v)$ for $\delta \to 0$ reads [16]

$$\sigma_k^{g+1}(r, \delta, v) = \sigma_k^g(r) + \frac{\delta}{2} \sigma_k^g(v) \partial_v \omega_v^g(r) + O(\delta^2) \tag{5.28}$$

where $\sigma_k^g(r)$ is the normalized differential of the second kind defined on $\mathcal{S}_g$ with pole at infinity of order $2k+2$, $\sigma_k^g(v) = \frac{\sigma_k^g(r)}{dr}|_{r=v}$ and $\partial_v = \dfrac{\partial}{\partial v}$. The differential $\omega_v^g(r)$ is the normalized Abelian differential of the third kind with poles at the points $Q^{\pm}(v) = (v, \pm\mu(v))$ with residue ± 1 respectively. The explicit expression of $\omega_v^g(r)$ has been given in (2.14) The differential $O(\delta^2)/\delta^2$ has a pole at $r = v$ of order 4 and zero residue.

From (5.28) we can get the expansion of the polynomial $P_k^{g+1}(r, \delta, v) = \tilde{\mu}(r) \dfrac{\sigma_k^{g+1}(r, \delta, v)}{dr}$, namely

$$P_k^{g+1}(r, \delta, v) = (r - v) P_k^g(r) + \frac{\delta}{2} \sigma_k^g(v) \left(\mu'(v) - (r - v) \sum_{k=1}^{g} r^{g-k} N_k'(v) \right) + O(\delta^2), \tag{5.29}$$

where $P_k^g(r)$ has been defined in (2.7) and the $N_k(v)$'s have been defined in (2.18). From (5.29) we

can evaluate the following

$$\left.\frac{P_k^{g+1}(r,\delta)}{P_0^{g+1}(r,\delta)}\right|_{r=v\pm\sqrt{\delta}} = \frac{P_k^g(v)}{P_0^g(v)} \pm \sqrt{\delta}\partial_v\left(\frac{P_k^g(v)}{P_0^g(v)}\right) + O(\delta). \tag{5.30}$$

To evaluate the first two equations in (5.27) at the point of phase transition we need also the following relations

$$\begin{aligned} q_{g+1}^{g+1}(v,v,\vec{u}) &= \Psi^g(v;\vec{u}) \\ \left.\frac{\partial}{\partial\tilde{u}_i}q_{g+1}^{g+1}(\tilde{u}_1,\tilde{u}_2,\vec{u})\right|_{\tilde{u}_1=\tilde{u}_2} &= \frac{1}{2}\frac{\partial}{\partial v}\Psi^g(v;\vec{u}), \quad i=1,2, \\ \left.\frac{\partial^2}{\partial\tilde{u}_1\partial\tilde{u}_2}q_{g+1}^{g+1}(\tilde{u}_1,\tilde{u}_2,\vec{u})\right|_{\tilde{u}_1=\tilde{u}_2} &= \frac{1}{4}\frac{\partial^2}{\partial v^2}\Psi^g(v;\vec{u}), \end{aligned} \tag{5.31}$$

where the function $\Psi^g(r;\vec{u})$ has been defined in (5.7). From (4.29-4.30), (5.30) and (5.31) we obtain

$$\begin{aligned} \left.\frac{R^{g+1}(\tilde{u}_i,\delta)}{P_0^{g+1}(\tilde{u}_i,\delta)}\right|_{\tilde{u}_1=\tilde{u}_2} &= \frac{2\mu^2(v)\partial_v\Psi^g(v;\vec{u})}{P_0^g(v)} + \Psi^g(v;\vec{u})\sum_{l=1}^{g+1}(2l-1)\tilde{\Gamma}^g_{g+1-l}\frac{P_{l-1}^g(v)}{P_0^g(v)} \\ &+ \sum_{k=1}^{g} q_k^g(\vec{u})\sum_{l=1}^{k}(2l-1)\tilde{\Gamma}^g_{k-l}\frac{P_{l-1}^g(v)}{P_0^g(v)}, \quad i=1,2. \end{aligned} \tag{5.32}$$

From (2.21), (5.10) and using the definition of $\Phi^g(r;\vec{u})$ in (5.8) the above relation can be written in the form

$$\left.\frac{R^{g+1}(\tilde{u}_i,\delta)}{P_0^{g+1}(\tilde{u}_i,\delta)}\right|_{\tilde{u}_1=\tilde{u}_2} = \frac{2\mu^2(v)\Phi^g(v;\vec{u}) + R^g(v,\vec{u})}{P_0^g(v)}, \quad i=1,2, \tag{5.33}$$

where $R^g(r,\vec{u})$ has been defined in (5.3). We need to consider also the quantity

$$\begin{aligned} &\left[\frac{1}{\tilde{u}_1-\tilde{u}_2}\left(\frac{R^{g+1}(\tilde{u}_1,\delta)}{P_0^{g+1}(\tilde{u}_1,\delta)} - \frac{R^{g+1}(\tilde{u}_2,\delta)}{P_0^{g+1}(\tilde{u}_2,\delta)}\right)\right]_{\tilde{u}_1=\tilde{u}_2} = 8\left.\left(\sqrt{\delta}\frac{\mu^2(v+\sqrt{\delta})}{P_0^{g+1}(v+\sqrt{\delta},\delta)}\partial_{\tilde{u}_1}\partial_{\tilde{u}_2}q_{g+1}^{g+1}(\tilde{u}_1,\tilde{u}_2,\vec{u})\right)\right|_{\delta=0} \\ &+2\left.\left(\left(\frac{\mu^2(v+\sqrt{\delta})}{P_0^{g+1}(v+\sqrt{\delta},\delta)} - \frac{\mu^2(v-\sqrt{\delta})}{P_0^{g+1}(v-\sqrt{\delta},\delta)}\right)\partial_{\tilde{u}_2}q_{g+1}^{g+1}(\tilde{u}_1,\tilde{u}_2,\vec{u})\right)\right|_{\delta=0} \\ &+\sum_{k=1}^{g}q_k^g(\vec{u})\sum_{l=1}^{k}(2l-1)\tilde{\Gamma}^g_{k-l}(\vec{u})\left.\left(\frac{1}{2\sqrt{\delta}}\left(\frac{P_{l-1}^{g+1}(v+\sqrt{\delta},\delta)}{P_0^{g+1}(v+\sqrt{\delta},\delta)} - \frac{P_{l-1}^{g+1}(v-\sqrt{\delta},\delta)}{P_0^{g+1}(v-\sqrt{\delta},\delta)}\right)\right)\right|_{\delta=0} \\ &+q_{g+1}^{g+1}(v,v,\vec{u})\sum_{l=1}^{g+1}(2l-1)\tilde{\Gamma}^g_{g+1-l}(\vec{u})\left.\left(\frac{1}{2\sqrt{\delta}}\left(\frac{P_{l-1}^{g+1}(v+\sqrt{\delta},\delta)}{P_0^{g+1}(v+\sqrt{\delta},\delta)} - \frac{P_{l-1}^{g+1}(v-\sqrt{\delta},\delta)}{P_0^{g+1}(v-\sqrt{\delta},\delta)}\right)\right)\right|_{\delta=0} \end{aligned}$$

where we have used (4.29-4.30) to obtain the right hand side. Using (2.21), (5.10), (5.30) and (5.31) the above reduces to the form

$$\left[\frac{1}{\tilde{u}_1-\tilde{u}_2}\left(\frac{R^{g+1}(\tilde{u}_1,\delta)}{P_0^{g+1}(\tilde{u}_1,\delta)} - \frac{R^{g+1}(\tilde{u}_2,\delta)}{P_0^{g+1}(\tilde{u}_2,\delta)}\right)\right]_{\tilde{u}_1=\tilde{u}_2} = \frac{\partial}{\partial_v}\left(\frac{2\mu^2(v)\Phi^g(v) + R^g(v)}{P_0^g(v)}\right). \tag{5.34}$$

From (5.30), (5.33) and (5.34), the system (5.27) on the point of phase transition $\tilde{u}_1 = \tilde{u}_2 = v$ reads

$$\begin{cases} 0 & = & \dfrac{\partial}{\partial v}\left(\dfrac{-12tP_1^g(v) + R^g(v,\vec{u}) + 2\mu^2(v)\Phi^g(v;\vec{u})}{P_0^g(v,\vec{u})}\right) \\ \\ x & = & \dfrac{-12tP_1^g(v,\vec{u}) + R^g(v,\vec{u}) + 2\mu^2(v)\Phi^g(v;\vec{u})}{P_0^g(v,\vec{u})} \\ \\ x & = & \left[-12t\dfrac{P_1^g(r,\vec{u})}{P_0^g(r,\vec{u})} + \dfrac{R^g(r)}{P_0^g(r,\vec{u})}\right]_{r=u_i}, \quad i = 1,\dots,2g+1 \end{cases} \tag{5.35}$$

From proposition 3.3 the last $2g+1$ equations in (5.35) are equivalent to the condition

$$-xP_0^g(r) - 12tP_1^g(r) + R^g(r) \equiv 0\,, \quad g > 0.$$

Therefore system (5.35) is equivalent to system (5.14) for $g > 0$. For $g = 0$ substituting (3.2) in (5.35) we obtain

$$\begin{cases} 0 = \partial_v(-6tu + f(u) + 2(v-u)(\Phi_0(v,u) - 6t)) \\ \\ x = -6tu + f(u) + 2(v-u)(\Phi_0(v,u) - 6t) \\ \\ x = -6tu + f(u) \end{cases} \tag{5.36}$$

where $u_1 = u$ and $f(u)$ is the initial data. It is clear that (5.36) is equivalent to (5.14) for $g = 0$.

$\square$

Proof of Theorem 5.6.

For getting the equations determining the trailing edge we have to repeat all the above calculation in a slightly different way. We write the $(g+1)$-phase solution for the variables $\tilde{u}_1 = v+\sqrt{\delta}$, $\tilde{u}_2 = v-\sqrt{\delta}$, $u_1 > u_2 > \cdots > u_{2g+1}$ in the form

$$\begin{cases} 0 & = & \dfrac{-1}{\log(\frac{\tilde{u}_1-\tilde{u}_2}{2})^2(\tilde{u}_1-\tilde{u}_2)}\left(12t\left(\dfrac{P_1^{g+1}(\tilde{u}_2,\delta)}{P_0^{g+1}(\tilde{u}_2,\delta)} - \dfrac{P_1^{g+1}(\tilde{u}_1,\delta)}{P_0^{g+1}(\tilde{u}_1,\delta)}\right) - \dfrac{R^{g+1}(\tilde{u}_1,\delta)}{P_0^{g+1}(\tilde{u}_1,\delta)} + \dfrac{R^{g+1}(\tilde{u}_2,\delta)}{P_0^{g+1}(\tilde{u}_2,\delta)}\right), \\ \\ x & = & \left[-12t\dfrac{P_1^{g+1}(r,\delta)}{P_0^{g+1}(r,\delta)} + \dfrac{R^{g+1}(r,\delta)}{P_0^{g+1}(r,\delta)}\right]_{r=\tilde{u}_2}, \\ \\ x & = & \left[-12t\dfrac{P_1^{g+1}(r,\delta)}{P_0^{g+1}(r,\delta)} + \dfrac{R^{g+1}(r,\delta)}{P_0^{g+1}(r,\delta)}\right]_{r=u_i}, \quad i = 1,\dots,2g+1 \end{cases} \tag{5.37}$$

The reason to write the hodograph transform in the form (5.37) is that system (5.37) is non degenerate at the phase transition $\tilde{u}_1 = \tilde{u}_2 = v$ when $v \in (u_{2j}, u_{2j-1})$, $\;1 \le j \le g+1$, $\;u_{g+2} = a$.

From Theorem 4.2 and proposition 5.2 we deduce that the last $2g+1$ equations in (5.37) in the limit $\tilde{u}_1 = \tilde{u}_2$ reduce to those of system (5.15).
Next we investigate the behavior of the Abelian differentials of the second kind $\sigma_k^{g+1}(r,\delta,v)$, $k \geq 0$, in the limit $\delta \to 0$, when v belongs to one of the gaps (5.13). We have that [16]

$$\sigma_k^{g+1}(r,\delta,v) \simeq \sigma_k^g(r) - \frac{1}{\log\delta}\,\omega_v^g(r)\int_{Q^-(v)}^{Q^+(v)} \sigma_k^g(\xi), \tag{5.38}$$

where $\omega_v^g(r)$ has been defined in (2.14) and $Q^{\pm}(v) = (v, \pm\mu(v))$. From (5.38) we can obtain the expansion for the polynomial $P_k^{g+1}(r,\delta,v) = \tilde{\mu}(r)\dfrac{\sigma_k^{g+1}(r,\delta,v)}{dr}$, namely

$$P_k^{g+1}(r,\delta,v) \simeq (r-v)P_k^g(r) - \frac{1}{\log\delta}\,(r-v)\mu(r)\omega_v^g(r)\int_{Q^-(v)}^{Q^+(v)} \sigma_k^g(\xi), \tag{5.39}$$

so that

$$\left.\frac{P_k^{g+1}(r,\delta)}{P_0^{g+1}(r,\delta)}\right|_{r=v\pm\sqrt{\delta}} \simeq \frac{\int_{Q^-(v)}^{Q^+(v)}\sigma_k^g(\xi)}{\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)}\left(1 \mp \sqrt{\delta}\log\delta\left(\frac{\sigma_k^g(v)}{\int_{Q^-(v)}^{Q^+(v)}\sigma_k^g(\xi)} - \frac{\sigma_0^g(v)}{\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)}\right)\right) \tag{5.40}$$

Using (5.3), (4.29-4.30) and (5.40) we have that

$$\begin{aligned}\left.\frac{R^{g+1}(\tilde{u}_i,\delta)}{P_0^{g+1}(\tilde{u}_i,\delta)}\right|_{\tilde{u}_1=\tilde{u}_2} =& \sum_{k=1}^{g} q_k^g(\vec{u})\sum_{l=1}^{k}(2l-1)\tilde{\Gamma}_{k-l}^g(\vec{u})\frac{\int_{Q^-(v)}^{Q^+(v)}\sigma_k^g(\xi)}{\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)} \\ &+ q_{g+1}^{g+1}(v,v,\vec{u})\sum_{l=1}^{g+1}(2l-1)\tilde{\Gamma}_{g+1-l}^g(\vec{u})\frac{\int_{Q^-(v)}^{Q^+(v)}\sigma_k^g(\xi)}{\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)},\ i=1,2.\end{aligned} \tag{5.41}$$

Adding and subtracting

$$\frac{2}{\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)}\sum_{i=1}^{2g+1}\int_{Q^-(v)}^{Q^+(v)}\frac{\mu(\xi)d\xi}{\xi-u_i}\partial_{u_i}q_g(\vec{u}),$$

to (5.41) and using (2.21) and (5.10) we obtain

$$\left.\frac{R^{g+1}(\tilde{u}_i,\delta)}{P_0^{g+1}(\tilde{u}_i,\delta)}\right|_{\tilde{u}_1=\tilde{u}_2} = \frac{1}{\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)}\int_{Q^-(v)}^{Q^+(v)}\frac{2\mu^2(\xi)\Phi^g(\xi;\vec{u})+R^g(\xi)}{\mu(\xi)}d\xi,\quad i=1,2 \tag{5.42}$$

where $\Phi^g(r;\vec{u})$ and $R^g(r)$ have been defined in (5.8) and (5.3) respectively. We observe that

$$\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi) \neq 0$$

for all v belonging to the gaps (5.13).

We need to consider also the quantity in the first equation of (5.37) namely

$$\left(\frac{-1}{\log(\frac{\tilde{u}_1-\tilde{u}_2}{2})^2(\tilde{u}_1-\tilde{u}_2)}\left(\frac{R^{g+1}(\tilde{u}_1,\delta)}{P_0^{g+1}(\tilde{u}_1,\delta)}-\frac{R^{g+1}(\tilde{u}_2,\delta)}{P_0^{g+1}(\tilde{u}_2,\delta)}\right)\right)\Bigg|_{\tilde{u}_1=\tilde{u}_2}=$$
$$-8\left(\frac{\sqrt{\delta}}{\log\delta}\frac{\mu^2(v+\sqrt{\delta})}{P_0^{g+1}(v+\sqrt{\delta},\delta)}\partial_{\tilde{u}_1}\partial_{\tilde{u}_2}q_{g+1}^{g+1}(\tilde{u}_1,\tilde{u}_2,\vec{u})\right)\Bigg|_{\delta=0}$$
$$-2\left(\left(\frac{\mu^2(v+\sqrt{\delta})}{\log\delta\, P_0^{g+1}(v+\sqrt{\delta},\delta)}+\frac{\mu^2(v-\sqrt{\delta})}{\log\delta\, P_0^{g+1}(v-\sqrt{\delta},\delta)}\right)\partial_{\tilde{u}_2}q_{g+1}^{g+1}(\tilde{u}_1,\tilde{u}_2,\vec{u})\right)\Bigg|_{\delta=0}$$
$$-\sum_{k=1}^{g}q_k^g(\vec{u})\sum_{l=1}^{k}(2l-1)\tilde{\Gamma}_{k-l}^g(\vec{u})\left(\frac{1}{2\sqrt{\delta}\log\delta}\left(\frac{P_{l-1}^{g+1}(v+\sqrt{\delta},\delta)}{P_0^{g+1}(v+\sqrt{\delta},\delta)}-\frac{P_{l-1}^{g+1}(v-\sqrt{\delta},\delta)}{P_0^{g+1}(v-\sqrt{\delta},\delta)}\right)\right)\Bigg|_{\delta=0}$$
$$-q_{g+1}^{g+1}(v,v,\vec{u})\sum_{l=1}^{g+1}(2l-1)\tilde{\Gamma}_{g+1-l}^g(\vec{u})\left(\frac{1}{2\sqrt{\delta}\log\delta}\left(\frac{P_{l-1}^{g+1}(v+\sqrt{\delta},\delta)}{P_0^{g+1}(v+\sqrt{\delta},\delta)}-\frac{P_{l-1}^{g+1}(v-\sqrt{\delta},\delta)}{P_0^{g+1}(v-\sqrt{\delta},\delta)}\right)\right)\Bigg|_{\delta=0}$$

Using (2.21), (5.10), (5.40) and (5.31) we simplify the above relation to the form

$$\left(\frac{-1}{\log(\frac{\tilde{u}_1-\tilde{u}_2}{2})^2(\tilde{u}_1-\tilde{u}_2)}\left(\frac{R^{g+1}(\tilde{u}_1,\delta)}{P_0^{g+1}(\tilde{u}_1,\delta)}-\frac{R^{g+1}(\tilde{u}_2,\delta)}{P_0^{g+1}(\tilde{u}_2,\delta)}\right)\right)\Bigg|_{\tilde{u}_1=\tilde{u}_2}=\frac{1}{\mu(v)\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)}\times$$
$$\left(2\mu^2(v)\partial_v\Psi^g(v;\vec{u})+\sum_{k=1}^{g}q_k^g(\vec{u})\sum_{l=1}^{k}(2l-1)\tilde{\Gamma}_{k-l}^g(\vec{u})\left(P_{l-1}^g(v)-P_0^g(v)\frac{\int_{Q^-(v)}^{Q^+(v)}\sigma_{l-1}^g(\xi)}{\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)}\right)\right.$$
$$\left.+\left(2\mu(v)\mu'(v)-4\mu(v)\frac{P_0^g(v,\vec{u})}{\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)}\right)\Psi^g(v;\vec{u})\right)$$

Using repeatedly (5.10) we can write the above relation in the form

$$\left(\frac{-1}{\log(\frac{\tilde{u}_1-\tilde{u}_2}{2})^2(\tilde{u}_1-\tilde{u}_2)}\left(\frac{R^{g+1}(\tilde{u}_1,\delta)}{P_0^{g+1}(\tilde{u}_1,\delta)}-\frac{R^{g+1}(\tilde{u}_2,\delta)}{P_0^{g+1}(\tilde{u}_2,\delta)}\right)\right)\Bigg|_{\tilde{u}_1=\tilde{u}_2}=\frac{1}{\mu(v)\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)}\times$$
$$\left(2\mu^2(v)\Phi^g(v;\vec{u})+R^g(v)-\frac{P_0^g(v)}{\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)}\int_{Q^-(v)}^{Q^+(v)}\frac{2\mu^2(\xi)\Phi^g(\xi;\vec{u})+R^g(\xi)}{\mu(\xi)}d\xi\right), \tag{5.43}$$

where $\Phi^g(r;\vec{u})$ and $R^g(r)$ have been defined in (5.8) and (5.3) respectively.

From (5.40), (5.42) and (5.43) system (5.37) can be reduced to the form

$$\begin{cases} 0 = 2\mu^2(v)\Phi^g(v;\vec{u}) + Z^g(v) + \dfrac{P_0^g(v)}{\int_{Q^-(v)}^{Q^+(v)}\sigma_0^g(\xi)}\displaystyle\int_{Q^-(v)}^{Q^+(v)} \frac{2\mu^2(\xi)\Phi^g(\xi;\vec{u}) + Z^g(\xi)}{\mu(\xi)}d\xi \\ 0 = \displaystyle\int_{Q^-(v)}^{Q^+(v)} \frac{2\mu^2(\xi)\Phi^g(\xi;\vec{u}) + Z^g(\xi)}{\mu(\xi)}d\xi \\ x = \left[-t\dfrac{P_1^g(r)}{P_0^g(r)} + \dfrac{R^g(r)}{P_0^g(r)}\right]_{r=u_i}, \quad i = 1,\dots,2g+1 \end{cases} \tag{5.44}$$

where the polynomial $Z^g(r)$ has been defined in (5.2). From proposition 3.3 the last $2g+1$ equations in (5.44) are equivalent to the condition

$$Z^g(r) \equiv 0\,, \;\; g > 0.$$

Therefore system (5.44) is equivalent for $g > 0$ to the system

$$\begin{cases} 0 = \Phi^g(v;\vec{u}) \\ 0 = \displaystyle\int_{Q^-(v)}^{Q^+(v)} \mu(\xi)\Phi^g(\xi;\vec{u})d\xi \\ x = \left[-t\dfrac{P_1^g(r)}{P_0^g(r)} + \dfrac{R^g(r)}{P_0^g(r)}\right]_{r=u_i}, \quad i = 1,\dots,2g+1. \end{cases} \tag{5.45}$$

Because of (5.11), when $v \in (u_{2j}, u_{2j-1}), 1 \le j \le g+1,\; u_{2g+2} = a$, we can split the integral of the second equation of (5.45) in the form

$$\begin{aligned}\int_{Q^-(v)}^{Q^+(v)} \mu(\xi)\Phi^g(\xi;\vec{u})d\xi &= \int_{Q^-(v)}^{u_{2j-1}} \mu(\xi)\Phi^g(\xi;\vec{u})d\xi + \int_{u_{2j-1}}^{Q^+(v)} \mu(\xi)\Phi^g(\xi;\vec{u})d\xi \\ &= 2\int_{u_{2j-1}}^{Q^+(v)} \mu(\xi)\Phi^g(\xi;\vec{u})d\xi.\end{aligned}$$

Therefore (5.45) is equivalent to (5.15) for $g > 0$.
For $g = 0$ substituting (3.2) in (5.44) it is easy to check that we obtain a system equivalent to (5.15).
□

Proof of Theorem 5.7.
The equations describing the point of gradient catastrophe of the g-phase solution can be obtained either considering the limit of the $(g+1)-$phase solution when three Riemann invariants coalesce, or

supposing that one of the $2g+1$ distinct Riemann invariants of the g-phase solution has a vertical inflection point for $t>0$. For proving Theorem 5.7 we follow the latter possibility.
On the solution of (3.6) $\partial_x u_l(x,t) = (\partial_{u_l}(-\lambda_l t + w_l))^{-1}$ [6] therefore a point of gradient catastrophe of the g-phase solution is determined by the system

$$\begin{cases} \partial_{u_l}(-\lambda_l t + w_l) = 0 \\ \\ (\partial_{u_l})^2(-\lambda_l t + w_l) = 0 \\ \\ x = \left[-t\dfrac{P_1^g(r)}{P_0^g(r)} + \dfrac{R^g(r)}{P_0^g(r)} \right]_{r=u_i}, \quad i = 1,\dots,2g+1 \end{cases} \tag{5.46}$$

where $1 \le l \le 2g+1$. We show that system (5.46) is equivalent to system (5.16). For the purpose we compute explicitly the derivative $\partial_{u_l}(-\lambda_l t + w_l)$. From a generalization of a result in [9] we obtain:

$$\frac{\partial}{\partial u_l}\frac{P_k^g(u_l)}{P_0^g(u_l)} = \frac{1}{2}\,\frac{\partial}{\partial r}\frac{P_k^g(r)}{P_0^g(r)}\bigg|_{r=u_l}, \tag{5.47}$$

so that

$$\begin{aligned} &\frac{\partial}{\partial u_l}(-\lambda_l(\vec{u})t + w_l(\vec{u})) = -6t\frac{\partial}{\partial r}\,\frac{P_1^g(r)}{P_0^g(r)}\bigg|_{r=u_l} + \frac{\sum_{\substack{k=1\\k\neq l}}^{2g+1}\prod_{\substack{m=1\\m\neq k,l}}^{2g+1}(u_l-u_m)}{P_0^g(u_l)}\partial_{u_l}q_g(\vec{u}) \\ &+\frac{\prod_{\substack{m=1\\m\neq l}}^{2g+1}(u_l-u_m)}{P_0^g(u_l)}(\partial_{u_l})^2 q_g(\vec{u}) - \frac{\prod_{\substack{m=1\\m\neq l}}^{2g+1}(u_l-u_m)}{(P_0^g(u_l))^2}\partial_{u_l}q_g(\vec{u})\partial_{u_l}P_0^g(u_l) \\ &+\frac{1}{2}\sum_{n=1}^{g}(2n-1)\frac{\partial}{\partial r}\,\frac{P_{l-1}^g(r)}{P_0^g(r)}\bigg|_{r=u_l}\sum_{m=n}^{g}q_m(\vec{u})\tilde{\Gamma}_{m-n} + \sum_{n=1}^{g}(2n-1)\frac{P_{n-1}^g(u_l)}{P_0^g(u_l)}\sum_{m=n}^{g}\partial_{u_l}(q_m(\vec{u})\tilde{\Gamma}_{m-n}). \end{aligned} \tag{5.48}$$

In the above relation we need to compute the derivative

$$\partial_{u_l}P_0^g(u_l) = \partial_r P_0^g(r)|_{r=u_l} + \partial_{u_l}P_0^g(r)|_{r=u_l},$$

where $P_0^g(r) = r^g + \alpha_1^0 r^{g-1} + \cdots + \alpha_g^0$. For computing the derivatives of the normalization constants $\alpha_1^0, \alpha_2^0, \dots, \alpha_g^0$ in $P_0^g(r)$ we need the following proposition.

Proposition 5.15 *[14] Let $\omega_1(r)$ and $\omega_2(r)$ two normalized Abelian differentials on $\mathcal{S}_g$. Let be $\xi = \dfrac{1}{\sqrt{r}}$ the local coordinate at infinity and*

$$\omega_1 = \sum_k a_k^1 \xi^k d\xi, \quad \omega_2 = \sum_k a_k^2 \xi^k d\xi.$$

Define the bilinear product

$$V_{\omega_1\omega_2} = \sum_{k\ge 0}\frac{a_{-k-2}^1 a_k^2}{k+1},$$

then

$$\frac{\partial}{\partial u_i}V_{\omega_1\omega_2} = \operatorname*{Res}_{r=u_i}\frac{\omega_1(r)\omega_2(r)}{dr}, \quad i=1,\dots,2g+1, \tag{5.49}$$

where $\mathrm{Res}_{r=u_i}\frac{\omega_1(r)\omega_2(r)}{dr}$ *is the residue of the differential* $\frac{\omega_1(r)\omega_2(r)}{dr}$ *evaluated at* $r=u_i$.

Applying the above proposition to σ_0 and σ_k, $k=0,\dots g-1$ we obtain after non trivial simplifications

$$\begin{aligned}\frac{\partial}{\partial u_l}\begin{pmatrix}\alpha_1^0\\ \alpha_2^0\\ \dots\\ \alpha_{g-1}^0\\ \alpha_g^0\end{pmatrix} = &-\frac{1}{2}\begin{pmatrix}1\\ u_l\\ \dots\\ u_l^{g-1}\\ u_l^{g-1}\end{pmatrix}-\frac{1}{2}\begin{pmatrix}0&0&0&\dots&0\\ 1&0&0&\dots&0\\ \dots&\dots&\dots&\dots&\dots\\ u_l^{g-3}&u_l^{g-4}&\dots&0&0\\ u_l^{g-2}&u_l^{g-3}&\dots&1&0\end{pmatrix}\begin{pmatrix}\alpha_1^0\\ \alpha_2^0\\ \dots\\ \alpha_{g-1}^0\\ \alpha_g^0\end{pmatrix}\\ &+\frac{1}{2}\frac{P_0^g(u_l)}{\prod_{k=1,k\neq l}^{2g+1}(u_l-u_k)}\begin{pmatrix}\tilde\Gamma_0&0&0&\dots&0\\ \tilde\Gamma_1&\tilde\Gamma_0&0&\dots&0\\ \dots&\dots&\dots&\dots&\dots\\ \tilde\Gamma_{g-2}&\tilde\Gamma_{g-3}&\dots&\tilde\Gamma_0&0\\ \Gamma_{g-1}&\tilde\Gamma_{g-2}&\dots&\tilde\Gamma_1&\tilde\Gamma_0\end{pmatrix}\begin{pmatrix}1\\ 3P_1^g(u_l)\\ \dots\\ (2g-3)P_{g-2}^g(u_l)\\ (2g-1)P_{g-1}^g(u_l)\end{pmatrix}\end{aligned} \tag{5.50}$$

where the $\tilde\Gamma_k$'s have been defined in (2.19). From the above formula we obtain

$$\begin{aligned}\partial_{u_l}P_0^g(u_l) =&\partial_r P_0^g(r)|_{r=u_l}+\partial_{u_l}P_0^g(r)|_{r=u_l}\\ =&\frac{1}{2}\partial_r P_0^g(r)|_{r=u_l}+\frac{1}{2}\frac{P_0^g(u_l)}{\prod_{k=1,k\neq l}^{2g+1}(u_l-u_k)}\sum_{n=1}^{g}(2n-1)P_{n-1}^g(u_l)\sum_{m=n}^{g}u_l^m\tilde\Gamma_{m-n}.\end{aligned} \tag{5.51}$$

Using the relations (4.20), (4.24), (5.47) and (5.51) we simplify (5.48) to the form

$$\begin{aligned}\frac{\partial}{\partial u_l}(-\lambda_l(\vec u)t+w_l(\vec u)) = &-\frac{Z^g(u_l)}{2(P_0^g(u_l))^2}\partial_r P_0^g(r)|_{r=u_l}+\frac{\sum_{\substack{k=1\\k\neq l}}^{2g+1}\prod_{\substack{m=1\\m\neq k,l}}^{2g+1}(u_l-u_m)}{P_0^g(u_l)}\partial_{u_l}q_g(\vec u)\\ &+\frac{1}{2P_0^g(u_l)}\partial_r\left(\sum_{n=1}^{g}(2n-1)P_{n-1}^g(r)\sum_{m=n}^{g}q_m\tilde\Gamma_{m-n}-xP_0^g(r)-12tP_1^g(r)\right)\Bigg|_{r=u_l},\end{aligned} \tag{5.52}$$

where the polynomial $Z^g(r)$ has been defined in (5.2). Applying in the second term of (5.52) the relations (5.10) we can rewrite (5.52) in the compact form

$$\frac{\partial}{\partial u_l}(-\lambda_l(\vec u)t+w_l(\vec u)) = -\frac{\partial}{\partial r}\frac{Z^g(r)}{2P_0^g(r)}\Bigg|_{r=u_l}+\frac{\prod_{\substack{m=1\\m\neq l}}^{2g+1}(u_l-u_m)}{P_0^g(u_l)}\Phi^g(u_l;\vec u), \tag{5.53}$$

where $\Phi^g(r;\vec u)$ has been defined in (5.8). From proposition 3.3 the last $2g+1$ equations in (5.46) are equivalent to the condition

$$Z^g(r)\equiv 0\,,\quad g>0.$$

Therefore we can simplify (5.53) to the form

$$\frac{\partial}{\partial u_l}(-\lambda_l(\vec{u})t + w_l(\vec{u})) = \frac{\prod_{\substack{m=1\\m\neq l}}^{2g+1}(u_l - u_m)}{P_0^g(u_l)}\Phi^g(u_l;\vec{u}), \tag{5.54}$$

when $u_1 > u_2 > \cdots > u_{2g+1}$ satisfy the g-phase Whitham equations. As regarding the second derivative $\frac{\partial^2}{\partial u_l^2}(-\lambda_l(\vec{u})t + w_l(\vec{u}))$, from (5.54) we obtain

$$\frac{\partial^2}{\partial u_l^2}(-\lambda_l(\vec{u})t + w_l(\vec{u})) = \frac{\partial}{\partial u_l}\left(\frac{\prod_{\substack{m=1\\m\neq l}}^{2g+1}(u_l - u_m)}{P_0^g(u_l)}\right)\Phi^g(u_l;\vec{u}) + \frac{\prod_{\substack{m=1\\m\neq l}}^{2g+1}(u_l - u_m)}{P_0^g(u_l)}\partial_{u_l}\Phi^g(u_l;\vec{u}). \tag{5.55}$$

Observing that

$$\partial_{u_l}\Phi^g(u_l;\vec{u}) = \partial_r\Phi^g(r;\vec{u})|_{r=u_l} + \partial_{u_l}\Phi^g(r;\vec{u})|_{r=u_l}, \quad \partial_r\Phi^g(r;\vec{u})|_{r=u_l} = 2\partial_{u_l}\Phi^g(r;\vec{u})|_{r=u_l},$$

we obtain the relation $\partial_{u_l}\Phi^g(u_l;\vec{u}) = \frac{3}{2}\partial_r\Phi^g(r;\vec{u})|_{r=u_l}$. Therefore

$$\frac{\partial^2}{\partial u_l^2}(-\lambda_l(\vec{u})t + w_l(\vec{u})) = \frac{\partial}{\partial u_l}\left(\frac{\prod_{m=1,m\neq l}^{2g+1}(u_l - u_m)}{P_0^g(u_l)}\right)\Phi^g(u_l;\vec{u}) + \frac{3}{2}\frac{\prod_{\substack{m=1\\m\neq l}}^{2g+1}(u_l - u_m)}{P_0^g(u_l)}\partial_r\Phi^g(r;\vec{u})|_{r=u_l}. \tag{5.56}$$

From (5.54-5.56) and the fact that $\frac{\prod_{m=1,m\neq l}^{2g+1}(u_l - u_m)}{P_0(u_l)} \neq 0$ for $u_1 > u_2 > \cdots > u_{2g+1}$, it is clear that system (5.46) is equivalent to system (5.16). $\square$

We remark that to avoid higher order degeneracies in system (5.16), we impose the condition

$$(\partial_r)^2\Phi^g(r;\vec{u})_{r=u_l(x_c,t_c)} \neq 0. \tag{5.57}$$

Indeed we prove that the above condition guarantees that the genus of solution of the Whitham equations increases at most by one in the neighborhood of the point of gradient catastrophe. It is sufficient to show that the transition to genus $g+2$ does not occur. For the purpose let us suppose l even and let us consider the function

$$0 \neq (\partial_r)^2\Phi^g(r;\vec{u})_{r=u_l(x_c,t_c)} = \text{const } \times \Phi^{g+2}(u_l; u_1, \ldots, u_{l-1}, u_l, u_l, u_l, u_l, u_l, u_{l+1}, \ldots, u_{2g+1}). \tag{5.58}$$

By proposition 5.3 in order to have a genus $g+2$ solution in the neighborhood of the point of gradient catastrophe, the function

$$\Phi^{g+2}(r; u_1, \ldots, u_{l-1}, u_l + \epsilon_1, u_l + \epsilon_2, u_l + \epsilon_3, u_l + \epsilon_4, u_l + \epsilon_5, u_{l+1}, \ldots, u_{2g+1})$$

has to change sign in each of the intervals $(u_l+\epsilon_2, u_l+\epsilon_3)$ and $(u_l+\epsilon_4, u_l+\epsilon_5)$ (l even) for arbitrary small $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4 > \epsilon_5 > 0$. Because of (5.58)

$$\Phi^{g+2}(r; u_1, \dots, u_{l-1}, u_l+\epsilon_1, u_l+\epsilon_2, u_l+\epsilon_3, u_l+\epsilon_4, u_l+\epsilon_5, u_{l+1}, \dots, u_{2g+1}) \neq 0$$

for sufficiently small $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4 > \epsilon_5 > 0$ and for $r \in (u_l+\epsilon_2, u_l+\epsilon_5)$. Therefore the transition to genus $g+2$ does not occur. We can exclude transitions from genus g to genus $g+n$, $n > 2$, by perturbations arguments. Therefore it is legitimate to consider the point of gradient catastrophe that solve (5.16) and satisfies (5.57) as a point of the boundary between the domains D_g and D_{g+1}. We remark that in the genus $g = 0$ case, the condition (5.58) is not essential as illustrated by theorem (3.2)

Proof of Theorem (5.9)

A phase transition may occur between the zero-phase solution and the two-phase solution. As shown on Figure 5.5 we can have a double leading edge, a trailing-leading edge and a double trailing edge. There are also other types of boundary between the zero-phase solution and the two-phase solution which we call "point of gradient catastrophe & leading edge" and "point of gradient catastrophe & trailing edge". Multiple transitions may also occur between the g-phase solution and the $(g+2)$-phase

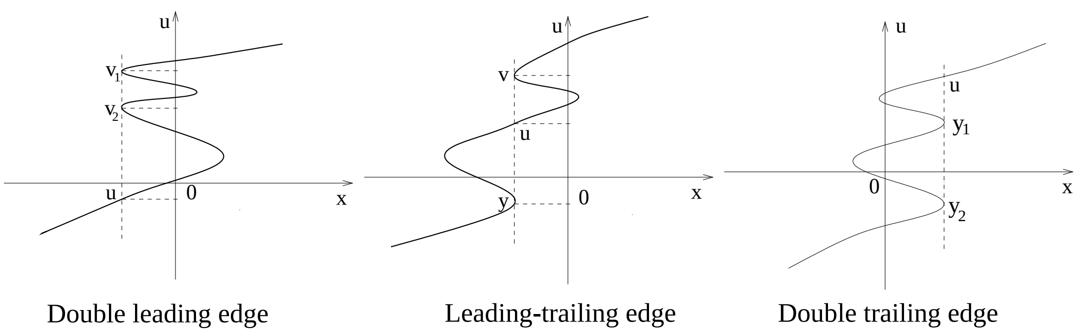

Figure 5.5: Three kinds of phase transition from the zero-phase solution to the two-phase solution

solution. In order to determine the systems which describe such phase transitions we consider the Riemann surface $\mathcal{S}_{g+2}$ of genus $g+2$ defined by

$$\tilde{\mu}^2 = (r - v_1 - \sqrt{\delta}_1)(r - v_1 + \sqrt{\delta}_1)(r - v_2 - \sqrt{\delta}_2)(r - v_2 + \sqrt{\delta}_2)\mu^2, \quad v \in \mathbb{R},$$

$$\mu^2(r) = \prod_{j=1}^{2g+1}(r - u_j), \quad u_1 > u_2 > \cdots > u_{2g+1},$$

where $v_j \neq u_i$, $j = 1, 2$, $i = 1, \dots, 2g+1$. Then we study the hodograph transform for the distinct variables $v_1 \pm \sqrt{\delta}_1$, $v_2 \pm \sqrt{\delta}_2$ and $u_1, \dots, u_{2g+1}$ in the *independent* limits $\delta_1 \to 0$ and $\delta_2 \to 0$. When

v_1 and v_2 belong to the bands (5.12), we are considering the double leading edge. Repeating the calculations done for the single leading edge we can determine the equations which describe the phase transition for the double leading edge, namely

$$\begin{cases} \partial_{v_1}\Phi^g(v_1;\vec{u})=0 \\ \Phi^g(v_1;\vec{u})=0 \\ \partial_{v_2}\Phi^g(v_2;\vec{u})=0 \\ \Phi^g(v_2;\vec{u})=0 \\ x=\left[-12t\dfrac{P_1^g(r)}{P_0^g(r)}+\dfrac{R^g(r)}{P_0^g(r)}\right]_{r=u_i} \qquad , \quad i=1,\dots,2g+1,\ g>0 \end{cases} \tag{5.59}$$

From (5.14-5.15) analogous systems can be obtained for the trailing-leading edge and double trailing edge.

For studying the phase transitions between the g-phase solution and the $(g+n)$-phase solution, $n\geq 1$, having n_1 leading edges and n_2 trailing edges, $n_1+n_2=n$ we consider the Riemann surface

$$\tilde{\mu}^2=\prod_{j=1}^{n_1}(r-v_j-\sqrt{\delta}_j)(r-v_j+\sqrt{\delta}_j)\prod_{k=1}^{n_2}(r-y_k-\sqrt{\rho}_k)(r-y_k+\sqrt{\rho}_k)\mu^2,$$

$$\mu^2(r)=\prod_{j=1}^{2g+1}(r-u_j),\quad u_1>u_2>\cdots>u_{2g+1}.$$

Here v_j, $j=1,\dots,n_1$, belongs to the bands (5.12) and $y_k,\ k=1\dots,n_2$, belongs to the gaps (5.13), $0<\delta_j\ll 1$, $j=1,\dots,n_1$, and $0<\rho_k\ll 1,\ k=1\dots,n_2$.

We study the hodograph transform (3.6) for the variables $v_j\pm\sqrt{\delta}_j,\ j=1\dots,n_1,\ y_k\pm\sqrt{\rho}_k$, $k=1\dots,n_2$ and $u_1,\dots,u_{2g+1}$ in the *independent* limits $\delta_j\to 0$, $j=1\dots,n_1$ and $\rho_k\to 0\ \ k=1\dots,n_2$. Repeating the calculations done for proving Theorem 5.4 and Theorem 5.6 it is easy to show that system (5.23) describes the phase transition between the g-phase solution and the $(g+n)$-phase solution having n_1 leading edges, n_2 trailing edges and no points of gradient catastrophe, $n_1+n_2=n>1$. If we suppose that the g-phase solution has also n_3 points of gradient catastrophe, $n_1+n_2+n_3=n$, then combining Theorem 5.4, Theorem 5.6 and Theorem 5.7 we obtain proposition 5.12. □

6 Conclusion

In this work we have constructed, in implicit form, the g-phase solution of the Whitham equations for monotone increasing smooth initial data $x=f(u)|_{t=0}$. The goal is obtained solving the Tsarev system (3.4). We have shown that the solution of the Tsarev system which satisfies the natural boundary conditions (3.7-3.10) is unique. Then we have investigated the conditions for the solvability of the hodograph transform (3.6). For the purpose we have derived all the equations which describe a phase

transition of the solution of the Whitham equations. Studying when phase transitions occur we have been able to prove the second main result of this work. Namely we have shown that when the initial data satisfies (5.1), the solution of the Whitham equations has genus $g \leq N$ for all x and $t \geq 0$. It is still an open problem to effectively determine, on the $x - t \geq 0$ plane, the function $0 \leq g(x,t) \leq N$ from the generic initial data $x = f(u)|_{t=0}$.

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